INTERNATIONAL JOURNAL OF CIRCUIT THEORY AND APPLICATIONS
Int. J. Circ. Theor. Appl. (2012)
Published online in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/cta.1796

A note on stability of linear time-variant electrical circuits having constant eigenvalues

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SUMMARY

The main aspect of this paper is to show that the stability of linear time-variant systems cannot be estimated from the location of the eigenvalues. For this purpose, two simple time-variant electrical circuits are presented, which have constant eigenvalues. As will be shown, the time-variant circuits can be asymptotically stable although there is a positive eigenvalue and this circuit can be unstable despite negative eigenvalues only. The idea behind is a suited time-variant state transformation of a linear time-invariant system. An electrical interpretation of both systems and of state transformations allows for an energetic evaluation from an electrical point of view even though the analytical solution is not necessarily known. Copyright © 2012 John Wiley & Sons, Ltd.

Received 14 December 2010; Revised 19 September 2011; Accepted 12 February 2012

KEY WORDS: state space model; linear time-variant system; eigenvalues; stability; electrical circuit; ordinary differential equation

1. INTRODUCTION

This paper is driven by the wrong but often heard presumption that stability of linear time-variant systems can be approximately assessed as in case of linear time-invariant systems, if the eigenvalues of the system matrix or the poles of the transfer function are varying slowly. Although there exists a counterexample in the literature on system theory [1], the example presented there is renowned as to be somewhat artificial. Still more, the success of some simulation results justifies this approach of determining stability of a linear time-variant system. The subject of this paper is not to condemn this treatment in total but rather to sensitize for problems that can occur. To this end, simple linear time-variant electrical circuits are given, where we cannot deduce any stability properties from the location of the eigenvalues—although the eigenvalues are indeed constant.

To inspect the stability behavior of linear time-variant systems, one can use the so-called dynamic eigenvalues (D-eigenvalues) (see [2–4] and [5,6], respectively). These dynamic eigenvalues are mean values of Lyapunov exponents and Floquet numbers [3,4,7] and can be computed by algorithms given in [2,8,9]. Moreover, it has been shown that for linear time-variant systems, the Ricatti equation takes the role of the characteristic equation [4,10]. Unfortunately, the Ricatti equation cannot be solved analytically and one has to use numerical solutions [11]. A further development is an LDU decomposition for a fundamental matrix of a linear time-variant system [12]. Canonical representations for single input and single output linear time-varying systems have been presented in [13]. Whereas these representations are signal flow diagrams containing integrators, an electrical circuit representation is presented here.

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The reader is assumed to be familiar with the basic concepts of linear time-variant systems and network theory (see, e.g., [14,15]). On the basis of these concepts, appropriate electrical circuits are synthesized to obtain insights into the stability issue. For this purpose, an electrical representation of a state space model is recapitulated in the next section. This approach has already been presented in [16], but this recapitulation is necessary because we use some modified notations here. Apart from that, this approach has been used for passive electrical circuits only, whereas we are also considering non-passive circuits. In Section 3, a time-variant state transformation of a linear time-invariant system is examined. The aim of this state transformation is the construction of appropriate electrical circuits. In Section 4, a linear time-variant circuit is presented, which is unstable despite constant negative eigenvalues. The converse case is presented in Section 5, where an electrical circuit is stable despite a constant positive eigenvalue. The subjacent state transformations are seized under energetic aspects in Section 5, and a conclusion is drawn in the last section.

2. ELECTRICAL CIRCUIT REPRESENTATION

In the literature on system theory. a Kirchhoff circuit is commonly viewed as an abstract real system having states, input, and output signals. They are collected in column vectors w, x, and y, respectively, where we assume n states, ℓ input, and m output signals. The equations describing the electrical circuit are partitioned into ordinary differential equations and algebraic equations. These state space equations

$$\mathcal{D}_t\{\boldsymbol{w}(t)\} = \hat{\boldsymbol{A}}\boldsymbol{w}(t) + \hat{\boldsymbol{B}}\boldsymbol{x}(t), \quad \text{with} \quad \boldsymbol{w}(t_0) = \boldsymbol{w}_0, \tag{1a}$$

$$\boldsymbol{y}(t) = \hat{\boldsymbol{C}}\boldsymbol{w}(t) + \hat{\boldsymbol{D}}\boldsymbol{x}(t) \tag{1b}$$

form the mathematical model, where $\mathcal{D}_t$ denotes a derivation with respect to time, (1a) is the state equation and (1b) is the output equation. The accents of the real system matrices $\hat{A}, \hat{B}, \hat{C}$, and $\hat{D}$ indicate that these matrices are generally time dependent. From a physical point of view, it is reasonable to assume bounded matrices

$$\left\|\hat{\boldsymbol{B}}\right\| < \infty, \quad \left\|\hat{\boldsymbol{C}}\right\| < \infty, \quad \text{and} \quad \left\|\hat{\boldsymbol{D}}\right\| < \infty, \tag{2}$$

to avoid nonnatural amplifications of the signals. To be brief, we assume a well-defined Kirchhoff circuit such that the state space equations have a unique solution.

2.1. *Electrical interpretation of a state space model*

An abstraction of a physical system by means of a mathematical model in form of the state space equation has the advantage of a generally feasible analysis, which is independent of the domain where the equations come from. But because of substitutions of quantities, we lose topological information, if we solely describe an electrical circuit by state space equations. Finding a corresponding electrical circuit can be a hard task, because we are now faced with a synthesis problem, whereas the setup of state space equations is merely a circuit analysis. For a first approach, we can utilize multi-ports [15] to obtain an electrical representation of the state space equations [16]; see Figure 1.

In this figure, the wires of the multi-ports are sketched with double lines to reflect the multidimensionality of currents and voltages. The inductances for realizing the time derivatives as well as all controlled sources are decoupled, whereas the negative system matrix is represented by a multi-port resistance. This electrical representation is of course somewhat artificial, but it is a first step toward an electrical circuit. As the algebraic output equations have no physical meaning, we concentrate on the synthesis of an electrical circuit for the state equation, which are represented on the left-hand side in Figure 1.

DOI: 10.1002/cta

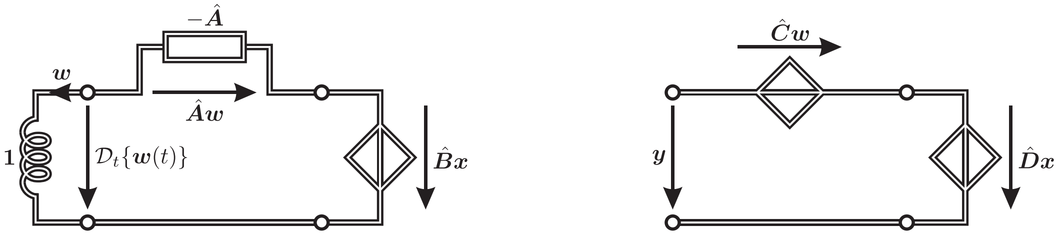

Figure 1. Electrical representation of the state equation (left) and output equation (right).

2.2. *Circuit synthesis*

Because of the choice of inductances for realizing the time derivatives, the states in Figure 1 act as electrical currents. For consistent physical units and a better association with electrical quantities, let us introduce

$$\boldsymbol{i}(t) = \boldsymbol{w}(t) \quad \text{and} \quad \boldsymbol{u}(t) = L\mathcal{D}_t\{\boldsymbol{w}(t)\}, \quad \text{with} \quad L > 0. \tag{3}$$

This results in an equivalent formulation of the state equation (1a),

$$L\mathcal{D}_t\{\boldsymbol{i}(t)\} + \hat{\boldsymbol{R}}\boldsymbol{i}(t) = \boldsymbol{v}_x(t), \quad \text{with} \quad \boldsymbol{i}(t_0) = \boldsymbol{i}_0, \tag{4a}$$

where $\hat{\boldsymbol{R}}$ denotes a resistance matrix and v_x is a vector of source voltages:

$$\hat{\boldsymbol{R}} = -L\hat{\boldsymbol{A}} \quad \text{and} \quad v_x(t) = L\hat{\boldsymbol{B}}\boldsymbol{x}(t). \tag{4b}$$

As it is known [16], these equations describe an externally passive electrical circuit, if and only if the symmetric part of $\hat{\boldsymbol{R}}$ is positive semidefinite. Unlike in [16], we do not stress passivity here, but rather we assume a symmetric part of the resistance matrix that is factorizable as follows:

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}_\mathrm{a} + \hat{\boldsymbol{R}}_\mathrm{s}, \quad \text{with} \quad \hat{\boldsymbol{R}}_\mathrm{a} = \hat{\boldsymbol{R}}_\mathrm{G} - \hat{\boldsymbol{R}}_\mathrm{G}^\mathrm{T} \quad \text{and} \quad \hat{\boldsymbol{R}}_\mathrm{s} = \hat{N}_\mathrm{s}^\mathrm{T}\hat{\boldsymbol{R}}_x\hat{N}_\mathrm{s} \geq 0. \tag{5}$$

The antimetric part $\hat{\boldsymbol{R}}_a$ is decomposed by the use of a strict lower triangular matrix $\hat{\boldsymbol{R}}_\mathrm{G}$. The remaining symmetric part $\hat{\boldsymbol{R}}_s$ is expressed by a product, which depends on a diagonal matrix $\hat{\boldsymbol{R}}_x$ and a lower triangular matrix $\hat{\boldsymbol{N}}_s$. Owing to a normalization, $\hat{\boldsymbol{N}}_s$ is assumed to have ones on its main diagonal. At last, we introduce

$$\boldsymbol{u}_x = \hat{\boldsymbol{N}}_\mathrm{s}^{-\mathrm{T}}\boldsymbol{v}_x, \tag{6}$$

where the upper $-\mathrm{T}$ denotes a transposition and an inversion of a matrix. Altogether, this yields the electrical circuit of Figure 2 being described by the following equations [16]:

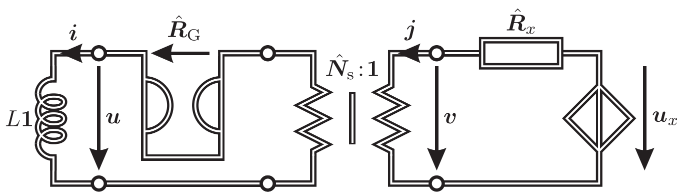

Figure 2. Electrical circuit for the state equation.

DOI: 10.1002/cta

$$\boldsymbol{u} = L\mathcal{D}_t\{\boldsymbol{i}\}, \quad \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{i} \end{bmatrix} = \begin{bmatrix} 1 & -\hat{\boldsymbol{R}}_\mathrm{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{N}}_\mathrm{s}^\mathrm{T} & 0 \\ 0 & \hat{\boldsymbol{N}}_\mathrm{s}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{j} \end{bmatrix}, \quad \boldsymbol{v} = \boldsymbol{u}_x - \hat{\boldsymbol{R}}_x \boldsymbol{j}. \tag{7}$$

The product of the two chain matrices mirrors the cascade of the depicted multi-port gyrator and multi-port transformer; see Figure 2.

Note that the inductances as well as the resistive sources are decoupled. An equivalent realization of this circuit is shown in Figure 3, where the multi-ports have been realized by usual gyrators and ideal transformers. The gyration resistances $\hat{r}_{\mu\nu}$ and turns ratios $\hat{n}_{\mu\nu}$ are elements of the matrices $\hat{\boldsymbol{R}}_G$ and $\hat{\boldsymbol{N}}_s$, respectively, whereas the resistances $\hat{r}_\nu$ are the main diagonal elements of $\hat{\boldsymbol{R}}_x$. Obviously, this circuit is (internally) passive, if and only if the resistances $\hat{r}_\nu$ are positive [16].

Now, a successive elimination of the vectors u, v, and j leads to

$$L\mathcal{D}_t\{i(t)\} + \left[\hat{\boldsymbol{R}}_\mathrm{G} - \hat{\boldsymbol{R}}_\mathrm{G}^\mathrm{T} + \hat{\boldsymbol{N}}_\mathrm{s}^\mathrm{T}\hat{\boldsymbol{R}}_x\hat{\boldsymbol{N}}_\mathrm{s}\right]\boldsymbol{i}(t) = \hat{\boldsymbol{N}}_\mathrm{s}^\mathrm{T}\boldsymbol{u}_x(t), \tag{8}$$

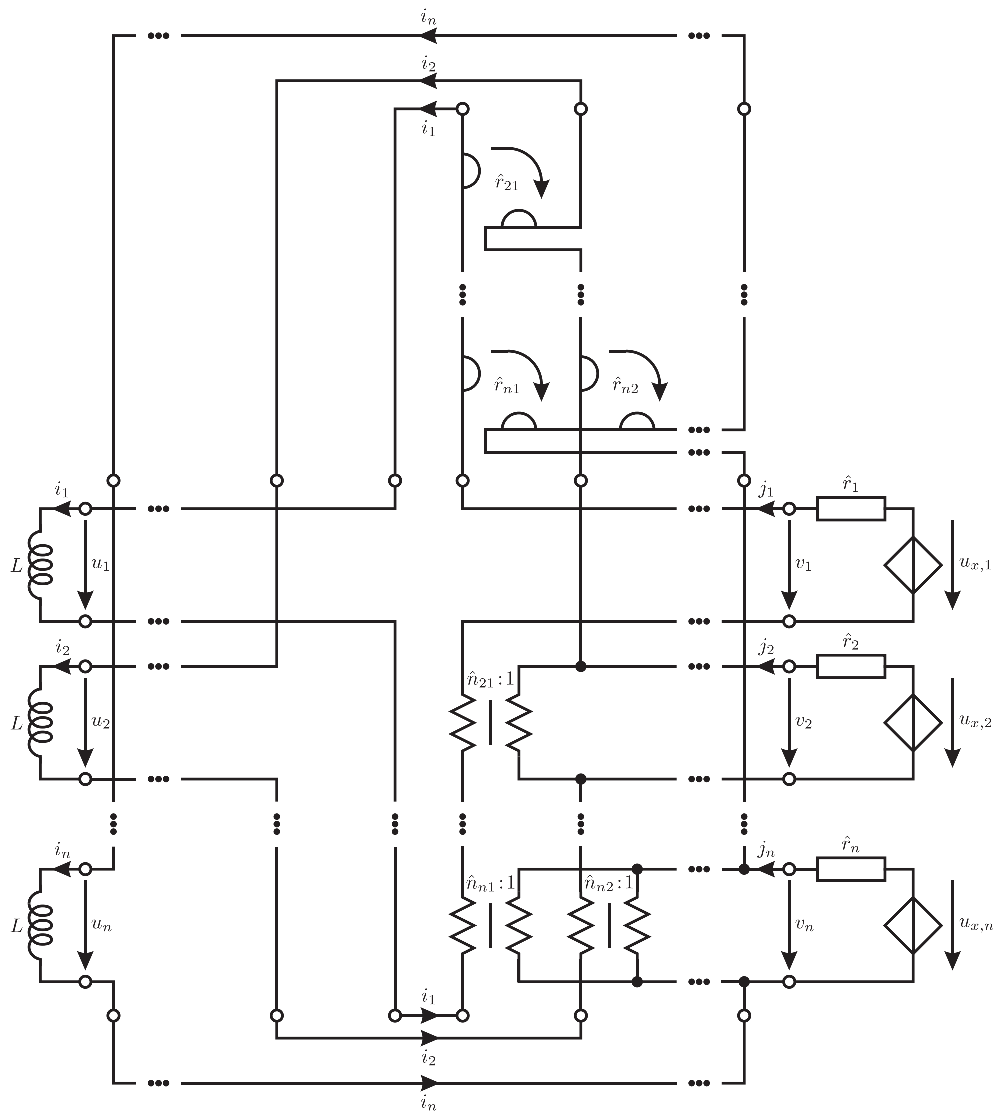

Figure 3. Explicit illustration of the circuit in Figure 2.

whereby the electrical circuit of Figure 3 obeys (4a). To sum up, the currents in this circuit are the states, and the state equation (1a) comprises the loop equations:

$$\boldsymbol{w}(t) = \boldsymbol{i}(t), \quad \hat{\boldsymbol{A}} = -\frac{1}{L}\left[\hat{\boldsymbol{R}}_{\mathrm{G}} - \hat{\boldsymbol{R}}_{\mathrm{G}}^{\mathrm{T}} + \hat{\boldsymbol{N}}_{\mathrm{s}}^{\mathrm{T}}\hat{\boldsymbol{R}}_x\hat{\boldsymbol{N}}_{\mathrm{s}}\right], \quad \text{and} \quad \hat{\boldsymbol{B}}x(t) = \frac{1}{L}\hat{\boldsymbol{N}}_{\mathrm{s}}^{\mathrm{T}}\boldsymbol{u}_x(t). \tag{9}$$

It has to be stressed that this circuit provides more information than the mathematical model, because it places topological information at the disposal and additionally gives insights into the energetic behavior.

3. A TIME-VARIANT STATE TRANSFORMATION

In the following section, we need linear time-variant systems with special properties. For this purpose, we inspect the effect of a time-variant state transformation on a linear time-invariant system. The latter obeys (1a) and (1b) with constant system matrices:

$$\mathcal{D}_t\{\boldsymbol{z}(t)\} = \boldsymbol{A}\boldsymbol{z}(t) + \boldsymbol{B}\boldsymbol{x}(t), \quad \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{z}(t) + \boldsymbol{D}\boldsymbol{x}(t), \quad \boldsymbol{z}(t_0) = \boldsymbol{z}_0. \tag{10}$$

To avoid confusion in the following discussion, the state vector has been renamed and we refrain from an accentuation of the constant system matrices. The solution of the initial value problem (10) reads

$$\boldsymbol{z}(t) = \boldsymbol{\Phi}(t-t_0)\boldsymbol{z}_0 + \int_{t_0}^{t}\boldsymbol{\Phi}(t-t')\boldsymbol{B}(t')x(t')\mathrm{d}t', \quad \boldsymbol{\Phi}(t) = \mathbf{exp}(\boldsymbol{A}t) = \sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}\boldsymbol{A}^\nu \tag{11}$$

for $t \geq t_0$. Here, $\boldsymbol{\Phi}$ is the state-transition matrix suited to investigate the stability of the system. The latter is stable (in the sense of Lyapunov), if

$$\|\boldsymbol{\Phi}(t)\| < \infty \quad \text{for} \quad t \geq 0 \tag{12}$$

holds. It is asymptotically stable if it is attractive in addition:

$$\lim_{t\to\infty}\|\boldsymbol{\Phi}(t)\| = 0. \tag{13}$$

As a consequence, the system is (asymptotically) stable, if all eigenvalues of the system matrix have a negative real part. Conversely, the system is unstable, if one of the eigenvalues has a positive real part.

Now, for the construction of a linear time-variant system, the state vector $\boldsymbol{z}$ is transformed by means of a regular transformation matrix $\hat{\boldsymbol{T}}(t)$:

$$\begin{bmatrix}\boldsymbol{w}\\ \mathcal{D}_t\{\boldsymbol{w}\}\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{T}} & 0\\ \mathcal{D}_t\{\hat{\boldsymbol{T}}\} & \hat{\boldsymbol{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{z}\\ \mathcal{D}_t\{\boldsymbol{z}\}\end{bmatrix} \tag{14a}$$

$$\Leftrightarrow \quad \begin{bmatrix}\boldsymbol{z}\\ \mathcal{D}_t\{\boldsymbol{z}\}\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{T}}^{-1} & 0\\ \mathcal{D}_t\{\hat{\boldsymbol{T}}^{-1}\} & \hat{\boldsymbol{T}}^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{w}\\ \mathcal{D}_t\{\boldsymbol{w}\}\end{bmatrix}, \tag{14b}$$

with $\mathcal{D}_t\{\hat{\boldsymbol{T}}^{-1}\} = -\hat{\boldsymbol{T}}^{-1}\mathcal{D}_t\{\hat{\boldsymbol{T}}\}\hat{\boldsymbol{T}}^{-1}$. This time-variant state transformation leads to a system that obeys the state equation (1a). The system matrices are given by

$$\hat{\boldsymbol{A}}(t) = \hat{\boldsymbol{T}}(t)[\boldsymbol{A} + \mathrm{j}\boldsymbol{\Omega}]\hat{\boldsymbol{T}}^{-1}(t), \quad \hat{\boldsymbol{B}}(t) = \hat{\boldsymbol{T}}(t)\boldsymbol{B}, \quad \hat{\boldsymbol{C}}(t) = \boldsymbol{C}\hat{\boldsymbol{T}}^{-1}(t), \quad \hat{\boldsymbol{D}}(t) = \boldsymbol{D}, \tag{15}$$

and the matrix $\boldsymbol{\Omega}$ depends on the underlying transformation matrix:

$$\mathrm{j}\boldsymbol{\Omega} = \hat{\boldsymbol{T}}^{-1}\mathcal{D}_t\{\hat{\boldsymbol{T}}\} \quad \Leftrightarrow \quad \mathcal{D}_t\{\hat{\boldsymbol{T}}\} = \hat{\boldsymbol{T}}\mathrm{j}\boldsymbol{\Omega}. \tag{16}$$

An electrical interpretation of the discussed transformation is shown in Figure 4. On the left-hand side, there is the linear time-invariant system, which is transformed to obtain the linear time-variant system of the right-hand side.

So far, the transformation matrix $\hat{\boldsymbol{T}}(t)$ is only limited to be regular. Thus, we can choose a constant matrix Ω and determine the associated matrix $\hat{\boldsymbol{T}}(t)$. With regard to (16), $\hat{\boldsymbol{T}}(t)$ can be interpreted as the transition matrix of a linear time-invariant system with the constant system matrix j $\boldsymbol{\Omega}$, where j denotes the imaginary unit:

$$\hat{\boldsymbol{T}}(t) = \exp(\mathrm{j}\boldsymbol{\Omega}t), \quad \text{with} \quad \mathcal{D}_t\{\hat{\boldsymbol{T}}(t)\} = \hat{\boldsymbol{T}}(t)\mathrm{j}\boldsymbol{\Omega}, \quad \hat{\boldsymbol{T}}(0) = 1, \quad \hat{\boldsymbol{T}}(-t) = \hat{\boldsymbol{T}}^{-1}(t). \tag{17}$$

By exploiting (11) and (14a) and (14b) in the absence of any excitation, one can find the transition matrix of the linear time-variant system:

$$\boldsymbol{w}(t) = \hat{\boldsymbol{\Phi}}(t, t_0)\boldsymbol{w}_0, \quad \text{with} \quad \hat{\boldsymbol{\Phi}}(t, t') = \hat{\boldsymbol{T}}(t)\hat{\boldsymbol{\Phi}}(t - t')\hat{\boldsymbol{T}}(-t'). \tag{18}$$

Similar to the time-invariant case, the system is stable, if

$$\left\|\hat{\boldsymbol{\Phi}}(t, t')\right\| < \infty \quad \text{for} \quad t \geq t' \tag{19}$$

holds. Especially, it is asymptotically stable, if the trajectory is additionally attractive:

$$\lim_{t\to\infty} \left\|\hat{\boldsymbol{\Phi}}(t, t')\right\| = 0 \quad \text{for all } t' \in \mathbb{R}. \tag{20}$$

The next two sections explore this state transformation for linear systems with two states only. For this purpose, we utilize a rotation matrix for the state transformation:

$$\mathbf{j}\boldsymbol{\Omega} = \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix} \quad \Leftrightarrow \quad \hat{T}(t) = \begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{bmatrix}, \quad \text{with} \quad \Omega > 0. \tag{21}$$

The resulting systems are generally represented by the electrical circuit of Figure 3, which is, for the sake of convenience, depicted in Figure 5. Note that this circuit represents the linear time-invariant as well as the linear time-variant system of Figure 4.

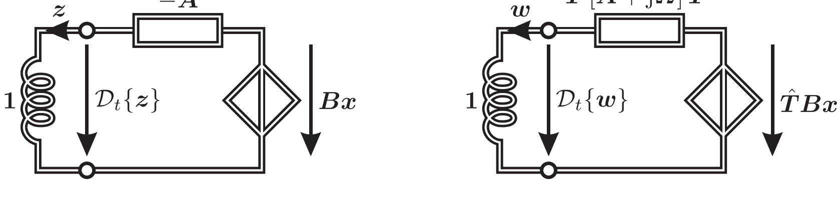

Figure 4. Electrical representation before (left) and after (right) a state transformation.

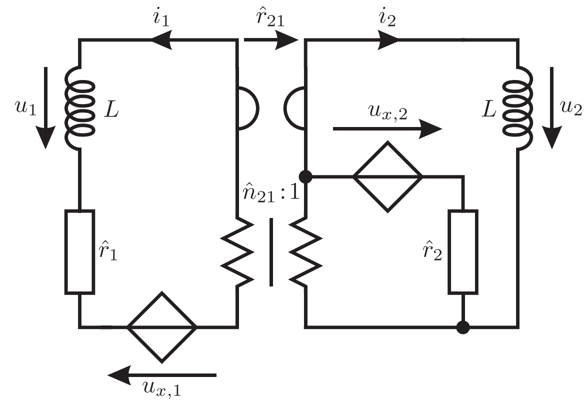

Figure 5. Electrical circuit of Figure 3 with two states.

4. INSTABILITY DESPITE CONSTANT NEGATIVE EIGENVALUES

To construct an unstable linear time-variant system with constant negative eigenvalues, we choose the matrix

$$\boldsymbol{A} + \mathrm{j}\boldsymbol{\Omega} = -\Omega \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \quad \text{with} \quad \alpha \in \mathbb{R}. \tag{22}$$

Both eigenvalues p_1 and p_2 of this system matrix are equal to $-\Omega$. Hence, they are negative independent of the parameter α. As the eigenvalues are invariant under a regular transformation [17], the time-variant system matrix of (28),

$$\hat{\boldsymbol{A}}(t) = -\Omega \begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix}, \tag{23}$$

has the same eigenvalues. Now, to accomplish instability of the time-variant system, we choose α such that the transformed time-invariant system with the system matrix

$$\boldsymbol{A} = -\Omega \begin{bmatrix} 1 & \alpha - 1 \\ 1 & 1 \end{bmatrix} \tag{24}$$

is unstable. The roots of the characteristic polynomial

$$\det(p\mathbf{1} - \boldsymbol{A}) = 0 \quad \Leftrightarrow \quad p^2 + 2p\Omega + [2 - \alpha]\Omega^2 = 0 \tag{25}$$

are the eigenvalues of A. With the choice $\alpha > 2$, the coefficients have one sign reversal, and because of Descartes' rule of signs, the polynomial has a positive root. Therefore, the time-variant system matrix

$$\hat{\boldsymbol{A}}(t) = -\Omega \begin{bmatrix} 1 - \frac{\alpha}{2}\sin(2\Omega t) & \frac{\alpha}{2} + \frac{\alpha}{2}\cos(2\Omega t) \\ -\frac{\alpha}{2} + \frac{\alpha}{2}\cos(2\Omega t) & 1 + \frac{\alpha}{2}\sin(2\Omega t) \end{bmatrix}, \quad \text{with} \quad \alpha > 2 \tag{26}$$

belongs to an unstable system, although its constant eigenvalues are negative. The solution of the unforced system can be deduced from (18):

$$\boldsymbol{w}(t) = \hat{\boldsymbol{\Phi}}(t, t_0)\boldsymbol{w}_0, \quad \text{with} \quad \hat{\boldsymbol{\Phi}}(t, t') = \hat{\boldsymbol{T}}(t)\boldsymbol{\Phi}(t - t')\hat{\boldsymbol{T}}(-t').$$

DOI: 10.1002/cta

It remains to determine the transition matrix of the linear time-invariant system, which can be obtained from an eigenvalue decomposition:

$$\Phi(t) = e^{-\Omega t}\begin{bmatrix} \cosh\left(\sqrt{\alpha-1}\Omega t\right) & -\sinh\left(\sqrt{\alpha-1}\Omega t\right)\sqrt{\alpha-1} \\ -\sinh\left(\sqrt{\alpha-1}\Omega t\right)/\sqrt{\alpha-1} & \cosh\left(\sqrt{\alpha-1}\Omega t\right) \end{bmatrix}, \alpha \neq 1. \tag{27}$$

An example for the resulting trajectory of the time-variant system is shown in Figure 6. Here, the values

$$\Omega = 1/s, \quad \alpha = 23/10, \quad t_0 = 0s, \quad w_0 = [1 \quad 0]^{\mathrm{T}}\mathrm{A}$$

have been chosen, where w_1 and w_2 are the first and second coordinate of the state vector w, respectively. Although both eigenvalues $p_1 = p_2 = -1/s$ of the time-variant system matrix $\hat{A}(t)$ are negative, the trajectory moves arbitrarily far away from the initial state and visualizes instability of the system.

For an electrical interpretation by means of the electrical circuit in Figure 5, one can find the corresponding element values in Table I.

5. STABILITY DESPITE A CONSTANT POSITIVE EIGENVALUE

The aim of this section is the construction of a time-variant system matrix that belongs to a stable system although it has a constant positive eigenvalue. To this end, we use the matrix

$$\boldsymbol{A} + \mathrm{j}\boldsymbol{\Omega} = -\begin{bmatrix} \sigma & \Omega \\ \Omega & \sigma \end{bmatrix}, \quad \text{with} \quad 0 < \sigma < \Omega, \tag{28}$$

having the eigenvalues $p_{1,2} = -\sigma \pm \Omega$, from which one is obviously positive. Because of the invariance of the eigenvalues with respect to a regular transformation, the eigenvalues are the same for the time-variant system matrix

$$\hat{\boldsymbol{A}}(t) = -\begin{bmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{bmatrix}\begin{bmatrix} \sigma & \Omega \\ \Omega & \sigma \end{bmatrix}\begin{bmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{bmatrix}. \tag{29}$$

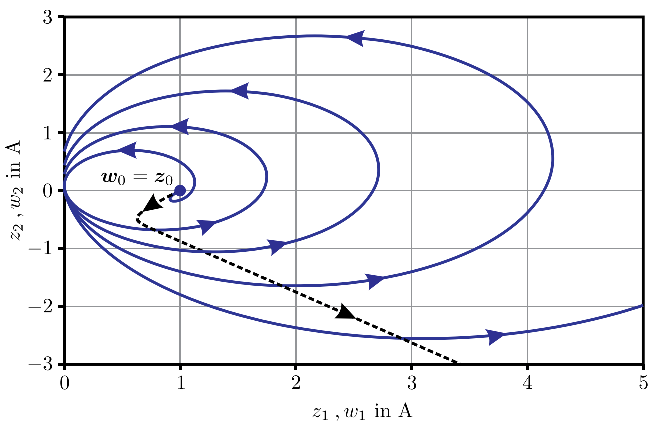

Figure 6. Trajectory of an unstable linear time-variant system with negative eigenvalues and the trajectory of the underlying linear time-invariant system (dashed line).

DOI: 10.1002/cta

Table I. Elements of the electrical circuit of the unstable linear time-variant system with negative eigenvalues and those of the linear time-invariant system.

System matrix	Element values of the associated circuit	
$A = -\Omega\begin{bmatrix}1 & \alpha-1\\ 1 & 1\end{bmatrix}$	$r_2 = L\Omega$ $r_1 = r_2\left[1-n_{21}^2\right]$	$n_{21} = \alpha/2$ $r_{21} = r_2[1-n_{21}]$
$\hat{A}(t) = -\Omega\begin{bmatrix}1-\frac{\alpha}{2}\sin(2\Omega t) & \frac{\alpha}{2}+\frac{\alpha}{2}\cos(2\Omega t)\\ -\frac{\alpha}{2}+\frac{\alpha}{2}\cos(2\Omega t) & 1+\frac{\alpha}{2}\sin(2\Omega t)\end{bmatrix}$	$\hat{r}_2 = r_2[1+n_{21}\sin(2\Omega t)]$ $\hat{r}_1\hat{r}_2 = r_1 r_2$	$\hat{n}_{21}\hat{r}_2 = n_{21}r_2\cos(2\Omega t)$ $\hat{r}_{21} = r_{21}-r_2$

The system matrix of the associated time-invariant system

$$\boldsymbol{A} = -\begin{bmatrix}\sigma & 0\\ 2\Omega & \sigma\end{bmatrix}, \quad \text{with} \quad \boldsymbol{A}^{\nu} = [-1]^{\nu}\begin{bmatrix}\sigma^{\nu} & 0\\ 2\Omega\nu\sigma^{\nu-1} & \sigma^{\nu}\end{bmatrix} \quad \text{for} \quad \nu\in\mathbb{N} \tag{30}$$

is of lower triangular form. As the eigenvalues of $\boldsymbol{A}$ are identical to the negative main diagonal elements, the transformed time-invariant system is asymptotically stable. Its transition matrix can be computed from (11) and has to be transformed according to (18) to attain the transition matrix of the time-variant system:

$$\hat{\boldsymbol{\Phi}}(t,t') = \hat{\boldsymbol{T}}(t)\boldsymbol{\Phi}(t-t')\hat{\boldsymbol{T}}(-t'), \quad \text{with} \quad \boldsymbol{\Phi}(t) = e^{-\sigma t}\begin{bmatrix}1 & 0\\ -2\Omega t & 1\end{bmatrix}.$$

For an estimation of the norm, it is advised to make use of the spectral norm

$$\left\|\hat{\boldsymbol{T}}\right\|_2 = \sup_{\|\boldsymbol{w}\|_2=1}\left\|\hat{\boldsymbol{T}}\boldsymbol{w}\right\|_2, \tag{31}$$

because it equals one for the orthogonal transformation matrix defined in (21):

$$\left\|\hat{\boldsymbol{\Phi}}(t,t')\right\|_2 \le \left\|\hat{\boldsymbol{T}}(t)\right\|_2 \|\boldsymbol{\Phi}(t-t')\|_2 \left\|\hat{\boldsymbol{T}}(-t')\right\|_2 = \|\boldsymbol{\Phi}(t-t')\|_2 < \infty. \tag{32}$$

Evidently, the asymptotic stability of the time-invariant system causes not only a bounded norm of the transition matrix $\hat{\boldsymbol{\Phi}}(t,t')$ but also attractivity:

$$\lim_{t\to\infty}\left\|\hat{\boldsymbol{\Phi}}(t,t')\right\|_2 \le \lim_{t\to\infty}\|\boldsymbol{\Phi}(t-t')\|_2 = 0 \quad \text{for all } t' \in\mathbb{R}. \tag{33}$$

The time-variant system with the system matrix

$$\hat{\boldsymbol{A}}(t) = -\begin{bmatrix}\sigma-\Omega\sin(2\Omega t) & \Omega\cos(2\Omega t)\\ \Omega\cos(2\Omega t) & \sigma+\Omega\sin(2\Omega t)\end{bmatrix}, \quad 0<\sigma<\Omega, \tag{34}$$

is therefore (asymptotically) stable despite the constant positive eigenvalue $\Omega-\sigma$. This is substantiated by the trajectory of Figure 7, on the basis of the specifications

$$\sigma = 0.5/\text{s}, \quad \Omega = 1/\text{s}, \quad t_0 = 0\text{s}, \quad \text{and} \quad \boldsymbol{w}_0 = [-1 \quad 0]^{\mathrm{T}}\mathbf{A}.$$

As it has been pointed out, the trajectory runs for $t\to\infty$ to the origin although it has the eigenvalues $p_1 = -1.5/\text{s}$ and $p_2 = 0.5/\text{s}$, from which the latter is positive. The element values of the corresponding electrical circuits are collected in Table II.

DOI: 10.1002/cta

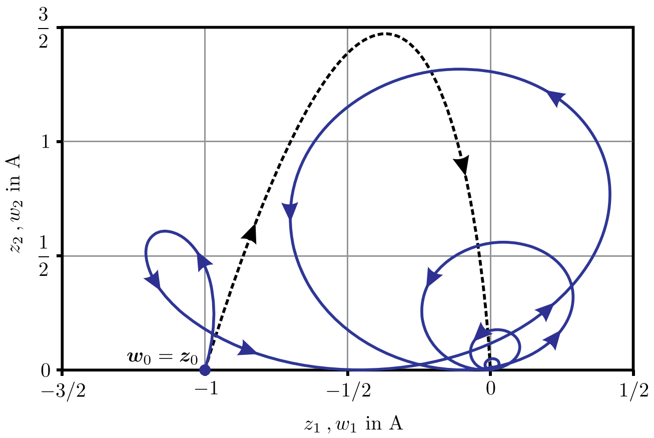

Figure 7. Trajectory of a stable linear time-variant system with a positive eigenvalue and the trajectory of the underlying linear time-invariant system (dashed line).

Table II. Elements of the electrical circuit of the stable linear time-variant system with a positive eigenvalue and those of the linear time-invariant system.

System matrix	Element values of the associated circuit	
$\mathbf{A} = -\begin{bmatrix} \sigma & 0 \\ 2\Omega & \sigma \end{bmatrix}$	$r_2 = L\sigma$ $r_1 = r_2\left[1 - n_{21}^2\right]$	$n_{21} = \Omega/\sigma$ $r_{21} = r_2 n_{21}$
$\hat{\mathbf{A}}(t) = -\begin{bmatrix} \sigma - \Omega\sin(2\Omega t) & \Omega\cos(2\Omega t) \\ \Omega\cos(2\Omega t) & \sigma + \Omega\sin(2\Omega t) \end{bmatrix}$	$\hat{r}_2 = r_2[1 + n_{21}\sin(2\Omega t)]$ $\hat{r}_1 = 2r_2 - \hat{r}_2\left[1 + n_{21}^2\right]$	$\hat{n}_{21}\hat{r}_2 = n_{21}r_2\cos(2\Omega t)$ $\hat{r}_{21} = 0$

6. PASSIVE STATE TRANSFORMATIONS

Finally, an electrical interpretation of state transformations is given. As we have observed in the last section, a state transformation of a stable system has led again to a stable system. A state transformation with this property is called a Lyapunov transformation. To investigate state transformations from an energetic point of view, let us assume an orthogonal transformation matrix $\hat{T}$ or, equivalently, a constant antimetric matrix $\boldsymbol{\Omega}$ as in (21):

$$\hat{\boldsymbol{T}}^{-1} = \hat{\boldsymbol{T}}^{\mathrm{T}} \quad \Leftrightarrow \quad \boldsymbol{\Omega} = -\boldsymbol{\Omega}^{\mathrm{T}}. \tag{35}$$

Under consideration of (17), the state transformation of (14a) and (14b) can be reformulated as

$$\begin{bmatrix} \boldsymbol{w} \\ \mathcal{D}_t\{\boldsymbol{w}\} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{T}} & 0 \\ 0 & \hat{\boldsymbol{T}}^{-\mathrm{T}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ j\boldsymbol{\Omega} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \mathcal{D}_t\{\boldsymbol{z}\} \end{bmatrix}, \tag{36}$$

where a product of two chain matrices occurs, which are associated with a multi-port transformer and a multi-port resistance; see Figure 8.

Accordingly, the state transformation is realized by a chain connection of a multi-port transformer and a multi-port resistance. The latter is lossless because of its (real) antimetric resistance matrix† and can be realized by a multi-port gyrator. In view of this lossless multi-port, we call this state transformation lossless. It has the special property to preserve the stored energy of the transformed system:

†Note that j $\boldsymbol{\Omega} = -A_a$ leads to a series connection with a symmetric resistance matrix $A + \mathrm{j}\,\boldsymbol{\Omega} = A_s$. This way, the constant multi-port gyrator for the realization of $-A_a$ can be removed by using a time-variant multi-port transformer.

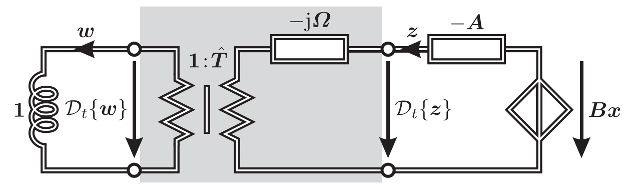

Figure 8. Electrical interpretation of the state transformation in Figure 4.

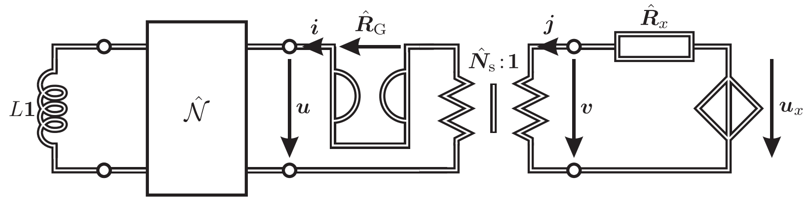

Figure 9. Electrical circuit of Figure 2 after a static state transformation.

$$E = \frac{1}{2}\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z} = \frac{1}{2}\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}.$$

As the energy function is also a Lyapunov function of this system, the observed conservation of stability properties in the examples is evident. Generally, we can transfer all attributes to a state transformation. It is called passive, lossless, static, and so on, if the corresponding electrical circuit has this property. This idea is visualized in Figure 9, where the result of a state transformation applied to the circuit in Figure 2 is shown.

7. CONCLUSIONS

In this paper, two simple circuits have been presented, which show that the location of the poles of a time-variant system matrix contains no information about the stability of the associated system: A linear time-variant system can be unstable, if its system matrix only has constant negative eigenvalues. On the contrary, it could even be asymptotically stable although it has a positive eigenvalue. Therewith, we cannot deduce stability of a linear time-variant system by arguing that the eigenvalues are slowly varying in the left complex half-plane.

Moreover, electrical interpretations for the state space model as well as for the state transformation have been given. These electrical circuits reduce the degree of abstraction considerably, and we gain deep insights into a system described by a system of linear differential equations of first order having time-variant system matrices. This concerns especially energetic aspects such as passivity and losslessness, which can be evaluated without knowing the exact solution. This way, the synthesized electrical circuits are comprehensive parametric representations of the underlying differential equations.

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