

Achievable and Crystallized Rate Regions of the Interference Channel with Interference as Noise

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Abstract—The interference channel achievable rate region is presented when the interference is treated as noise. The formulation starts with the 2–user channel, and then extends the results to the n –user case. The rate region is found to be the convex hull of the union of n power control rate regions, where each power control rate region is upperbounded by a $(n - 1)$ -dimensional hyper-surface characterized by having one of the transmitters transmitting at full power. The convex hull operation lends itself to a time-sharing operation depending on the convexity behavior of those hyper-surfaces. In order to know when to use time-sharing rather than power control, the paper studies the hyper-surfaces convexity behavior in details for the 2–user channel with specific results pertaining to the symmetric channel. It is observed that most of the achievable rate region can be covered by using simple On/Off binary power control in conjunction with time-sharing. The binary power control creates several corner points in the n –dimensional space. The crystallized rate region, named after its resulting crystal shape, is hence presented as the time-sharing convex hull imposed onto those corner points; thereby offering a viable new perspective of looking at the achievable rate region of the interference channel.

I. INTRODUCTION

One important communication model in wireless communication is the interference channel, which is subject to intensive research nowadays. For example, the model is relevant for cellular networks in which multiple base stations transmit data to their respective subscribers and thereby causing interference at the unintended receivers, and ad-hoc networks in which nodes are active at the same moment in the same frequency band. For a better understanding of the interference channel, it is crucial to know its capacity region, i.e., the maximum set of all achievable rate points. It serves also as a benchmark for the comparison of different schemes. Unfortunately, the capacity region of the 2–user interference channel has been

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an open problem for about 30 years [3], [4]. Information-theoretic bounds through achievable rate regions have been proposed, most famously with the Han-Kobayashi region [5]. The capacity of the Gaussian interference channel under strong interference has been found in [6], [7]. Recent results on the 2–user interference channel to within one bit of capacity have been shown in [8], where a simplified Han-Kobayashi scheme was used in which the message is split in two parts, the private part and the common part. The transmit signal is then a superposition of those two signals. By a smart allocation of power between those two parts it was shown that this scheme is asymptotically optimal using a new metric, which is referred to as the generalized degrees of freedom. The generalization of the obtained results to the n -user case is rather difficult. As such, for the n -user case mainly the capacity slope as a function of the Signal-to-Noise-Ratio (SNR) are known. It was shown in [9] that for very high SNR, the capacity can be approximated by $C = \frac{n}{2} \log(\text{SNR}) + o(\log(\text{SNR}))$, where the second term vanishes by definition for extremely high SNR.

The aforementioned referenced literature focused mainly on the 2–user interference channel from an information-theoretic point of view with highly sophisticated and thus quite complicated transmitters and receivers. There are other works in literature that tackle the practical issues in order to improve the performance of the interference channels. Power control is one element of critical importance. In [10], a framework for the uplink power control is constructed and iterative power control is proposed. Adaptive modulation and coding (AMC) can be combined with power control to enhance the network performance [11]. For multiple channel (such as OFDM) and multiple cell case, joint AMC and power control have been widely employed [12]. Beamforming and spatial diversity can also be utilized when communicating over the MIMO channel [13]. Interference avoidance [14] has also attracted many recent attentions. Finally, many distributed solutions are proposed [15]–[17] with the benefit of simple implementation or low data overhead.

In this paper, the achievable rate region is discussed for the n –user interference channel when the interference is treated as additive Gaussian white noise and no multi-user detection is employed. Examples where we encounter the need to define such rate region are found in multicell communications, in addition to mesh and sensor networks where the preference is to use low-complexity transceivers. It is also interesting to note that using the strategy of treating interference as additional noise proves to be asymptotically optimal, i.e., it was shown in [8] that treating interference as noise is optimal, as long as the interference power in dB is lower than half of the useful signal power given that the power to noise

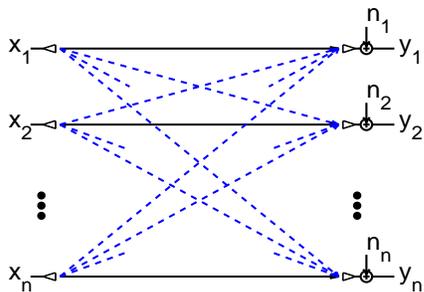


Fig. 1: n -user interference channel

ratio is asymptotically high. This result was extended to the non-asymptotic case independently by three research groups [18]–[20]. The generalization to the asymmetric case and the n -user interference channel is given in [18]–[20] as well, where it was shown that it is optimal to treat the interference as noise whenever a similar (sufficient) condition holds. The references [18]–[20] considered only the Gaussian interference channel, while the general discrete many-to-one and one-to-many memoryless channels were investigated in [21]. It was shown in [21] that treating interference as noise is also optimal in the discrete memoryless channel as long as the received signal at the interfered receiver is stochastically degraded compared to the received signals of the other receivers. The optimality of treating interference as noise for the multiple antenna case has been considered by [22]–[24].

This paper finds the achievable rate region for the n -user interference channel as the convex hull of the union of n rate regions formed via power control, where each rate region is upperbounded by a hyper-surface of dimension $n - 1$ characterized by having one of the transmitters operating at full power. Given that there is a convex hull operation imposed onto the hyper-surfaces, it is important to know their convexity behavior in order to determine when time-sharing should be applied. This is treated in details for the 2-user interference channel. As the convex hull operation lends itself naturally to a time-sharing operation, and based on the convexity conditions found, the paper discusses when a time-sharing strategy should be employed rather than pure power control, and then presents specific results pertaining to the 2-user symmetric channel. It is observed that the achievable rate region can be practically approximated by using simple On/Off binary power control in conjunction with time-sharing. The On/Off binary power control creates several corner points in the n -dimensional rate region, and employing a convex hull time-sharing operation on those points achieves what is denoted as a crystallized rate region.

The system setup is presented in section II. Section III discusses the achievable rate region for the 2-user interference channel, and then generalizes the results to the n -user case. Section IV focuses on characterizing the 2-user rate region in terms of convexity or concavity and when time-sharing is optimal with specific results to the symmetric channel. Section V introduces the concept of the crystallized rate region where time-sharing and On/Off binary power control are used. Finally, the conclusion is drawn in section VI.

II. SYSTEM MODEL

The n -user interference channel is presented in Fig.1 with n transmitters and n receivers. The i^{th} transmitter transmits its signal x_i to the intended i^{th} receiver with power P_i . The receivers have independent additive complex white Gaussian noise with zero mean and variance of σ_n^2 . Each transmitter is assumed to have a peak power constraint of P_{\max} . Each transmitter has a single antenna and communicates over a frequency flat channel. $g_{i,j}$ denotes the channel power gain received at the i^{th} receiver from the j^{th} transmitter. Therefore, $g_{i,i}$ is the channel gain of the i^{th} desired signal, whereas $g_{i,j}$ with $j \neq i$ represents the interfering channel gain. \mathbf{P} is the transmit power vector of length n , where the i^{th} element P_i denotes the transmit power of the i^{th} transmitter. The interference is treated as additive noise throughout this paper. R_i denotes the maximum reliable rate of communication for the i^{th} transmit-receive pair. Therefore, the achievable rate for the i^{th} transmit-receive pair is written as:

$$R_i(\mathbf{P}) = \log_2 \left(1 + \frac{g_{i,i}P_i}{\sigma_n^2 + \sum_{j \neq i} g_{i,j}P_j} \right). \quad (1)$$

The next section finds the achievable rate region for such n transmit-receive pairs.

III. ACHIEVABLE RATE REGION FRONTIERS FOR THE INTERFERENCE CHANNEL

First, the section considers the 2-user interference channel. The rate region problem is analyzed by formulating its underlying nonconvex power control problem, for which we find a closed form analytical solution. The rate region is described by finding the maximum possible data rates achievable when each user is subject to a maximum transmit power constraint. This section then introduces the 3-user case to study the effect of adding a new dimension; and finally, by induction, the result is generalized for the n -user case.

A. 2-user Achievable Rate Region Frontiers

In the case of the 2-user interference channel, Eq. (1) can be expressed as a function of P_1 and P_2 as $R_i(P_1, P_2)$, $i = 1, 2$. For notational brevity, the channel gains are normalized by the noise variance, specifically: $a = g_{1,1}/\sigma_n^2$, $b = g_{1,2}/\sigma_n^2$, $c = g_{2,2}/\sigma_n^2$, and $d = g_{2,1}/\sigma_n^2$. R_1 and R_2 can therefore be written as:

$$\begin{aligned} R_1(P_1, P_2) &= \log_2 \left(1 + \frac{aP_1}{1 + bP_2} \right), \\ R_2(P_1, P_2) &= \log_2 \left(1 + \frac{cP_2}{1 + dP_1} \right). \end{aligned} \quad (2)$$

For notational brevity, $\Phi(p_1, p_2)$ denotes a point in the rate region marked by having $P_1 = p_1$ and $P_2 = p_2$. Effectively, the x -coordinate of $\Phi(p_1, p_2)$ is $R_1(p_1, p_2) = r_1$, and the y -coordinate of $\Phi(p_1, p_2)$ is $R_2(p_1, p_2) = r_2$. The first objective is to find the achievable rate region frontiers of Eq. (2) through power control of P_1 and P_2 , where each transmitter is subject to the maximum power constraint of P_{\max} . The frontier herein denotes the line (or generally, the $(n - 1)$ -dimensional surface for the n -user channel) which traces the rate region via power control.

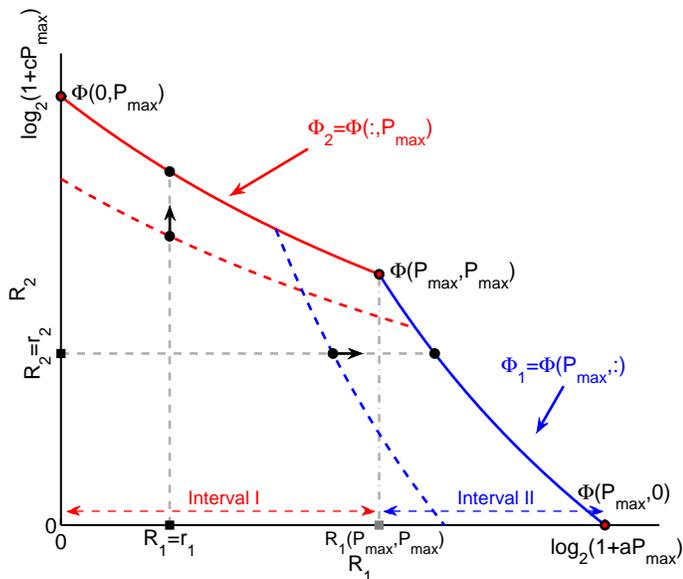


Fig. 2: 2-user power-control rate region

B. Rate Region Frontiers Formulation

The rate region frontier can be traced by setting R_1 to a certain value r_1 , and then by sweeping r_1 over its full possible range from 0 to $R_1(P_{\max}, 0)$ while finding the maximum R_2 value that can be achieved for each r_1 . From Eq. (2), R_1 is monotonically increasing in P_1 and monotonically decreasing in P_2 , thus point $R_1(P_{\max}, 0)$ corresponds to point $\Phi(P_{\max}, 0)$ on the x -axis in Fig. 2, representing the maximum value R_1 can attain. Similarly for the y -axis, the maximum value that R_2 can attain is $R_2(0, P_{\max})$, alternatively corresponding to point $\Phi(0, P_{\max})$. Those points represent the cases in which one of the users is silent, while the other is transmitting at full power. Similarly, point $\Phi(P_{\max}, P_{\max})$ has the coordinates of $R_1(P_{\max}, P_{\max})$ and $R_2(P_{\max}, P_{\max})$.

Hence, for a constant rate $R_1 = r_1$,

$$R_1(P_1, P_2) = r_1 = \log_2 \left(1 + \frac{aP_1}{1 + bP_2} \right). \quad (3)$$

Therefore, the relation between P_1 and P_2 is obtained as follows:

$$P_1 = \frac{1}{a}(1 + bP_2)(2^{r_1} - 1). \quad (4)$$

From Eq. (4), for a constant $R_1(P_1, P_2) = r_1$, $R_2(P_1, P_2)$ can now be written as a function of one parameter as $R_2(P_2)$, for $R_1 = r_1$, specifically:

$$R_2(P_2) = \log_2 \left(1 + \frac{cP_2}{1 + \frac{d}{a}(1 + bP_2)(2^{r_1} - 1)} \right). \quad (5)$$

It is important to analyze the behavior of $R_2(P_2)$ in terms of P_2 . This is presented in the following lemma:

Lemma 1. *Setting R_1 at a constant rate, $R_1(P_1, P_2) = r_1$, $R_2(P_2)$ is a monotonically increasing function in P_2 .*

Proof: The proof is provided in Appendix A. ■

Using this lemma, the following corollary of uniqueness property is obtained:

Corollary 1. *For every rate tuple (r_1^*, r_2^*) , there is a unique power tuple (p_1^*, p_2^*) .*

Proof: i) a direct implication of monotonicity in Eq. (5) is that if R_2 is equal to a constant r_2^* at the rate of $R_1 = r_1$, then there is a unique p_2^* that achieves r_2^* , ii) when p_2^* is determined, then $P_1 = p_1^*$ is uniquely defined from Eq. (4), iii) from p_1^* and p_2^* , R_1 is uniquely defined as $R_1 = r_1 = r_1^*$ from Eq. (3). Thus, p_1^* and p_2^* uniquely define a point in the rate region with coordinates r_1^* and r_2^* . ■

In other words, any point in the rate region is achieved solely by a unique power tuple. This leads to what we denote by *potential lines* Φ in the rate region, which are formed by holding one power dimension constant to a certain value and sweeping the other power dimension over its full range. In that regard, to describe a potential line marked by having P_1 held at a constant power P_{cst} , we use the following notation $\Phi(P_{\text{cst}}, :)$ to be equivalent to $\Phi(P_{\text{cst}}, p_2)$ where $P_1 = P_{\text{cst}}$ and $0 \leq p_2 \leq P_{\max}$. Based on the uniqueness property just discussed, we have the following corollary (illustrated in Fig.3):

Corollary 2. *Potential lines¹ along one power dimension do not intersect, i.e., $\Phi(:, p_2)$ and $\Phi(:, p_2')$ do not intersect if $p_2 \neq p_2'$.*

The problem of finding the power control rate region frontiers then simplifies into finding the maximum value $R_2(P_2)$ that can be achieved for any value of $R_1(P_1, P_2) = r_1$. Effectively, the formulation of the power control rate region is:

$$\begin{aligned} & \arg \max_{P_2} R_2(P_2) \\ & \text{subject to } R_1(P_1, P_2) = r_1, \\ & P_i \leq P_{\max}, \quad i = 1, 2. \end{aligned} \quad (6)$$

r_1 is swept over the full range of R_1 , i.e., $0 \leq r_1 \leq R_1(P_{\max}, 0)$. The power control optimization problem in Eq. (6) is not straightforward as it is a nonconvex problem [26]. However, by splitting the rate range of R_1 into two intervals: *Interval 1* for $0 \leq r_1 \leq R_1(P_{\max}, P_{\max})$ and *Interval 2* for $R_1(P_{\max}, P_{\max}) \leq r_1 \leq R_1(P_{\max}, 0)$, we are able to find a closed form analytical solution for each interval. The analysis of the optimization problem in Eq. (6) over these two intervals follows in the next two subsections.

C. R_2 Frontier for Interval 1: $0 \leq r_1 \leq R_1(P_{\max}, P_{\max})$

As Eq. (3) is monotonically increasing in P_1 and monotonically decreasing in P_2 , r_1 can only exceed $R_1(P_{\max}, P_{\max})$ when P_2 is less than P_{\max} . Thus, $P_2 = P_{\max}$ is attainable only when $0 \leq r_1 \leq R_1(P_{\max}, P_{\max})$, and P_2 needs to be less than P_{\max} otherwise. From the proof provided for Lemma 1, where Eq. (5) is proved to be monotonically increasing in P_2 , and for the following *Interval 1* range of r_1 :

¹The property in Corollary 2 is the reason for denoting these lines as *potential lines*, where the nomenclature is borrowed from electromagnetics based on a similar property for equipotential lines of an electric field [25].

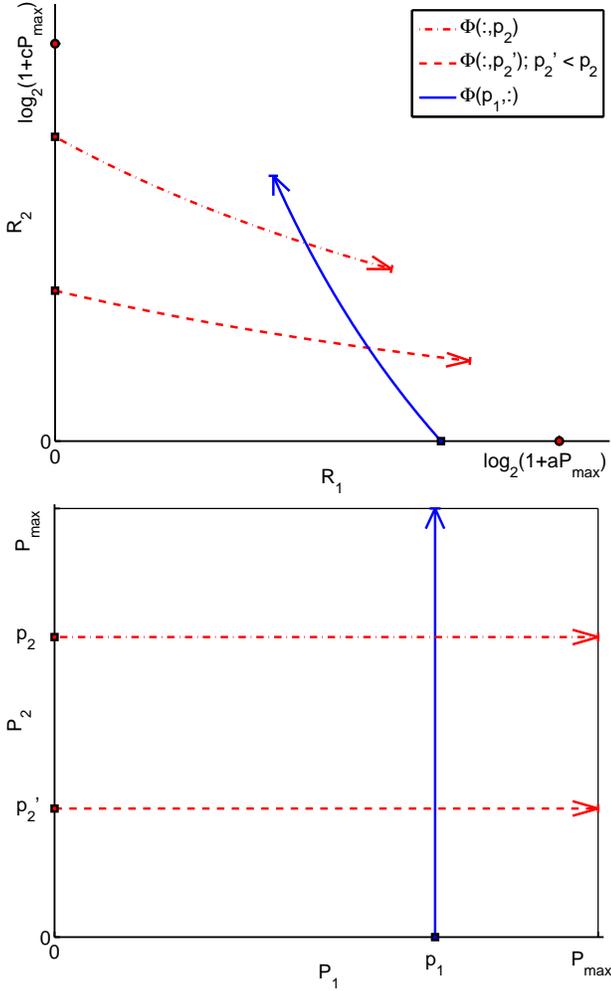


Fig. 3: potential lines illustration in rate region and power region

$0 \leq r_1 \leq R_1(P_{\max}, P_{\max})$, the solution to the optimization problem is:

$$\arg \max_{P_2} R_2(P_2) = P_{\max}. \quad (7)$$

Therefore, in this range of r_1 , using Eq. (4) and Eq. (7), R_2 is expressed by a function of r_1 as follows:

$$R_2(r_1) = \log_2 \left(1 + \frac{cP_{\max}}{1 + \frac{d}{a}(1 + bP_{\max})(2^{r_1} - 1)} \right). \quad (8)$$

Over this *Interval 1* range of r_1 , the relation found in Eq. (8) describes the expression governing the potential line $\Phi(:, P_{\max})$, in which P_2 is held at constant maximum power P_{\max} and P_1 sweeps its full range from 0 to P_{\max} . For brevity, the potential line $\Phi(:, P_{\max})$ is denoted as Φ_2 , where the second power dimension, P_2 , is held at the maximum power. In this range of r_1 , Φ_2 defined by Eq. (8) represents a power control frontier of the rate region as shown in Fig.2.

D. R_2 Frontier for Interval 2: $R_1(P_{\max}, P_{\max}) \leq r_1 \leq R_1(P_{\max}, 0)$

Using symmetry of the previous result, for a constant rate $R_2 = r_2$, there is a linear relation between P_1 and P_2 . Thus, $R_1(P_1, P_2)$ can be written in function of one parameter P_1 as follows:

$$R_1(P_1) = \log_2 \left(1 + \frac{aP_1}{1 + \frac{b}{c}(1 + dP_1)(2^{r_2} - 1)} \right). \quad (9)$$

By symmetry of the result in Lemma 1, $R_1(P_1)$ is monotonically increasing in P_1 . Thus, by symmetry, for the following range of r_2 : $0 \leq r_2 \leq R_2(P_{\max}, P_{\max})$, we have:

$$\arg \max_{P_1} R_1(P_1) = P_{\max}. \quad (10)$$

Basically, the value found in Eq.(10) describes the frontier for the following rate ranges of: $0 \leq r_2 \leq R_2(P_{\max}, P_{\max})$ and $R_1(P_{\max}, P_{\max}) \leq r_1 \leq R_1(P_{\max}, 0)$ – similar to the former subsection III-C where $P_2 = P_{\max}$ described the frontier for the following rate ranges of: $0 \leq r_1 \leq R_1(P_{\max}, P_{\max})$ and $R_2(P_{\max}, P_{\max}) \leq r_2 \leq R_2(P_{\max}, 0)$.

Therefore, the value of R_1 at the frontier is:

$$R_1(P_{\max}, P_2) = \log_2 \left(1 + \frac{aP_{\max}}{1 + bP_2} \right). \quad (11)$$

Hence, for this *Interval 2* range of r_1 : $R_1(P_{\max}, P_{\max}) \leq r_1 \leq R_1(P_{\max}, 0)$, the value of P_2 that achieves the frontier follows as:

$$P_2 = \frac{1}{b} \left(\frac{aP_{\max}}{2^{r_1} - 1} - 1 \right). \quad (12)$$

So effectively, the value found in Eq. (12) is the answer for the optimization problem in Eq. (6) for this range of R . Explicitly, for $R_1(P_{\max}, P_{\max}) \leq r_1 \leq R_1(P_{\max}, 0)$, we have:

$$\arg \max_{P_2} R_2(P_2) = \frac{1}{b} \left(\frac{aP_{\max}}{2^{r_1} - 1} - 1 \right). \quad (13)$$

Therefore, in this range of r_1 , using Eq. (12) and Eq. (5), R_2 is expressed in function of r_1 as follows:

$$R_2(r_1) = \log_2 \left(1 + \frac{\frac{c}{b}(aP_{\max} - (2^{r_1} - 1))}{(2^{r_1} - 1)(1 + dP_{\max})} \right). \quad (14)$$

The relation found in Eq. (14) describes the expression governing the potential line $\Phi(P_{\max}, :)$, where P_1 is held at a constant maximum power P_{\max} and P_2 sweeps its full range from 0 to P_{\max} . Similarly for brevity, the potential line $\Phi(P_{\max}, :)$ is denoted as Φ_1 . In this range of r_1 , Φ_1 as defined by Eq. (14) represents a power control frontier of the rate region as shown in Fig.2.

E. The 2–User Achievable Rate Region

This subsection consolidates the two earlier results to fully describe the rate region frontiers.

- For *Interval 1* of r_1 : $0 \leq r_1 \leq R_1(P_{\max}, P_{\max})$

$$\arg \max_{P_2} R_2(P_2) = P_{\max},$$

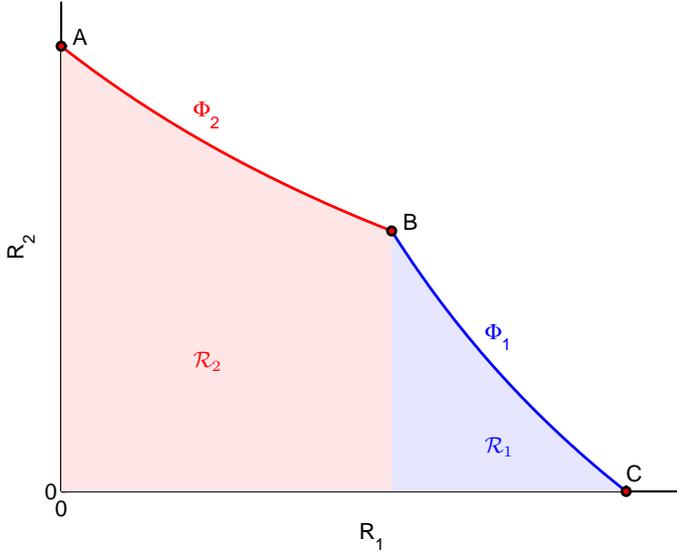


Fig. 4: rate regions $\mathcal{R}_1 = \mathcal{R}\{\Phi_1\}$ and $\mathcal{R}_2 = \mathcal{R}\{\Phi_2\}$

and the power control frontier, $\Phi_2 = \Phi(\cdot, P_{\max})$, is expressed as:

$$R_2(r_1) = \log_2 \left(1 + \frac{cP_{\max}}{1 + \frac{d}{a}(1 + bP_{\max})(2^{r_1} - 1)} \right). \quad (15)$$

Let $\mathcal{R}_2 = \mathcal{R}\{\Phi_2\}$ denotes the rate region outer-bounded by Φ_2 as shown in Fig. 4.

- For Interval 2 of r_1 : $R_1(P_{\max}, P_{\max}) \leq r_1 \leq R_1(P_{\max}, 0)$

$$\arg \max_{P_2} R_2(P_2) = \frac{1}{b} \left(\frac{aP_{\max}}{2^{r_1} - 1} - 1 \right),$$

and the power control frontier, $\Phi_1 = \Phi(P_{\max}, \cdot)$, is expressed as:

$$R_2(r_1) = \log_2 \left(1 + \frac{\frac{c}{b}(aP_{\max} - (2^{r_1} - 1))}{(2^{r_1} - 1)(1 + dP_{\max})} \right). \quad (16)$$

Similarly, let $\mathcal{R}_1 = \mathcal{R}\{\Phi_1\}$ denotes the rate region outer-bounded by Φ_1 as shown in Fig.4.

In Fig.4, point A denotes point $\Phi(0, P_{\max})$ (user 2 transmitting solely at full power), point B denotes point $\Phi(P_{\max}, P_{\max})$ (both users are transmitting simultaneously at full power), and point C denotes point $\Phi(P_{\max}, 0)$ (user 1 transmitting solely at full power).

The rate region for a 2-user interference channel achieved through power control is obtained as:

$$\mathcal{R}_1 \cup \mathcal{R}_2. \quad (17)$$

Finally, the 2-user rate region, denoted as \mathcal{R} , is found as the convex hull of the power control rate region. It is defined as:

$$\mathcal{R} = \text{Convex Hull}\{\mathcal{R}_1 \cup \mathcal{R}_2\}. \quad (18)$$

The treatment of the achievable rate region for the n -user interference channel follows next. It starts by considering a

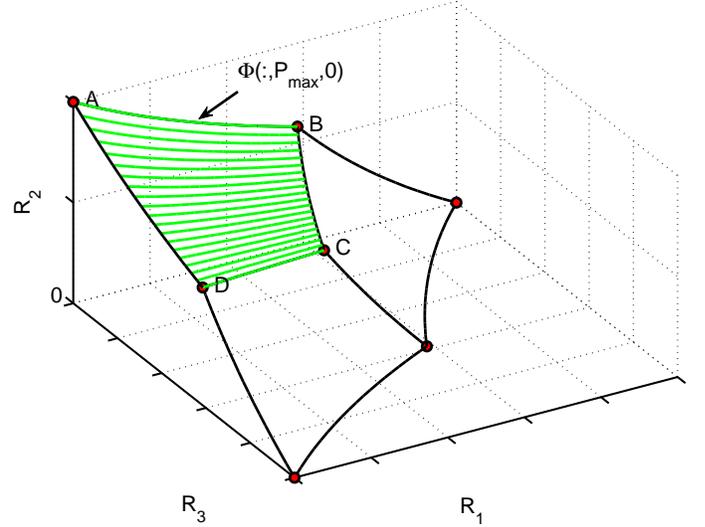


Fig. 5: 3-user interference channel achievable rate region

3-user interference channel to show the effect of adding a new dimension, and then generalizes the result for the n -user case.

F. 3-User Example: Effect of Increasing P_3 from 0 to P_{\max}

The rate region for the 3-user case is illustrated in Fig.5. The following notation of $\Phi(P_1, P_2, P_3)$ denotes a point in the rate region with coordinates of $[R_1(P_1, P_2, P_3), R_2(P_1, P_2, P_3), R_3(P_1, P_2, P_3)]$. Accordingly, $\Phi(\cdot, P_{\max}, P_3)$ describes a line characterized by sweeping the transmit power P_1 of the first transmitter from 0 to P_{\max} , with the second transmitter transmitting at P_{\max} and the third transmitter transmitting at a power value of P_3 . Similarly, $R_i(\cdot, P_{\max}, \cdot)$ represents a surface in the rate region marked by sweeping the full range of P_1 and P_3 , and holding P_2 at P_{\max} .

When $P_3 = 0$, the same setup and results that are described earlier in this section applies. Thus, for the rate range of $0 \leq r_1 \leq R_1(P_{\max}, P_{\max}, 0)$ and $0 \leq r_2 \leq R_2(0, P_{\max}, 0)$ and $R_3 = 0$, the frontier can be described as $\Phi(\cdot, P_{\max}, 0)$, which is the potential line from point A to point B in Fig. 5. As P_3 increases, the goal is to describe its effect and how it is traced in the rate region.

Revisiting Eq. (1), a fixed P_3 has the effect of just an additive noise term in $R_1(\mathbf{P})$ and $R_2(\mathbf{P})$. Hence, all the previous results in section III are applicable for any value of P_3 in describing the frontier for R_1 and R_2 , since the effect of P_3 can be lumped in the noise term. Thus, for the range of $0 \leq r_1 \leq R_1(P_{\max}, P_{\max}, P_3)$ and $0 \leq r_2 \leq R_2(0, P_{\max}, P_3)$, where P_3 is constant, the frontier line on R_1 and R_2 is $\Phi(\cdot, P_{\max}, P_3)$, i.e., characterized by having $P_2 = P_{\max}$. Consequently, the potential lines (or surfaces) concept in the 3-user case carries through.

Next, the frontier on R_3 is described. For each value of P_3 , $\Phi(\cdot, P_{\max}, P_3)$ traces one of the highlighted curves in Fig. 5. For the collection of lines to form a frontier, we want to prove that at each increasing value of P_3 these non-intersecting potential lines monotonically increase in the R_3 dimension. This is evident from the relation between R_3 and P_3 in Eq. (1).

The maximum value of R_3 that can be achieved in this case is when $P_3 = P_{\max}$, i.e., $R_3(\cdot, P_{\max}, P_{\max})$. Therefore, the highlighted frontier surface in Fig. 5 is the closed potential surface $\Phi(\cdot, P_{\max}, \cdot)$. The boundary contours of this surface are the potential lines: $A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, and $D \leftrightarrow A$, defined as $\Phi(\cdot, P_{\max}, 0)$, $\Phi(P_{\max}, P_{\max}, \cdot)$, $\Phi(\cdot, P_{\max}, P_{\max})$, and $\Phi(0, P_{\max}, \cdot)$, respectively.

By symmetry of interchanging P_1 , P_2 , and P_3 , the 3–user rate region is found via the convex hull onto the union of the regions bounded by these three surfaces: $\Phi(P_{\max}, \cdot, \cdot)$, $\Phi(\cdot, P_{\max}, \cdot)$, and $\Phi(\cdot, \cdot, P_{\max})$. The rate region \mathcal{R} is therefore expressed as: $\mathcal{R} = \text{Convex Hull}\{\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3\}$, where $\mathcal{R}_i = \mathcal{R}\{\Phi_i\}$ is the region outer-bounded by the potential surface Φ_i , where $\Phi_i = \Phi(\dots, P_i = P_{\max}, \dots)$ is the surface characterized by having P_{\max} in the i^{th} power position. (Note that the intersection of potential surfaces is a potential line, as two of the dimensional inputs become equal, i.e., $\Phi(P_{\max}, P_{\max}, \cdot) \in \mathcal{R}_1$ and $\Phi(P_{\max}, P_{\max}, \cdot) \in \mathcal{R}_2$.)

G. n –User Generalization

The case for n –user generalization is done by induction. For the n^{th} added dimension to the existing $n - 1$ dimensions problem, the additional power effect of P_n can be lumped in the additive noise term of the existing expressions, and thus the results for R_1, \dots, R_{n-1} hold and carry through. The potential hyper-surfaces for fixed P_n are non-intersecting and monotonically increasing in P_n , and thus the maximum outer limit is reached with $P_n = P_{\max}$ for the appropriate range of R_1, \dots, R_{n-1} . Invoking symmetry, we can generalize over all the rates ranges, therefore arriving to the following theorem.

Theorem 1. *The achievable rate region of the n –user interference channel by treating the interference as noise is:*

$$\mathcal{R} = \text{Convex Hull}\{\cup_{i=1}^n \mathcal{R}_i\}, \quad (19)$$

$\mathcal{R}_i = \mathcal{R}\{\Phi_i\}$, where Φ_i is a hyper-surface frontier of $n - 1$ dimensions, characterized by holding the i^{th} transmitter at full power.

Note that Theorem 1 also holds for different thermal noise levels or different maximum power levels.

IV. CONVEXITY CHARACTERISTICS OF THE POWER CONTROL FRONTIERS FOR THE 2–USER INTERFERENCE CHANNEL

This section focuses on the 2–user interference channel and studies the behavior of the power control frontiers, i.e., the potential lines Φ_1 and Φ_2 , in terms of convexity and concavity in order to determine when the convex hull operation entails employing time-sharing. This happens whenever any of the potential lines, or segment thereof, is convex, which enables higher data rate to be achieved using time-sharing rather than using power control. Furthermore, specific results pertaining to the symmetric channel are presented at the end of this section.

The power control frontiers equations for the 2–user case are:

- $\Phi_1: R_2(r_1) = \log_2 \left(1 + \frac{c}{b} \frac{(aP_{\max} - (2^{r_1} - 1))}{(2^{r_1} - 1)(1 + dP_{\max})} \right)$,

- $\Phi_2: R_2(r_1) = \log_2 \left(1 + \frac{cP_{\max}}{1 + \frac{d}{a}(1 + bP_{\max})(2^{r_1} - 1)} \right)$.

It is not clear when Φ_i is convex or concave, or whether it can exhibit a non-stationary inflection point. The non-stationary inflection point happens when the potential line has simultaneously a convex segment and a concave segment. The convexity behavior is thus treated next in more details.

A. Convexity or Concavity of the Power Control Frontiers

By using Eq. (4) when $P_2 = P_{\max}$, the potential line Φ_2 depends on P_1 through the following relation of r_1 and P_1 :

$$P_1 = \frac{1}{a}(1 + bP_{\max})(2^{r_1} - 1).$$

Therefore, the second derivative of Φ_2 with respect to r_1 leads to the following expression in function of P_1 :

$$\frac{\partial^2 \Phi_2}{\partial r_1^2} = (\alpha + adP_1)^2 - (a - \alpha)(a - \alpha + acP_{\max}), \quad (20)$$

where $\alpha = d + dbP_{\max}$.

If the potential line is concave (i.e. $\frac{\partial^2 \Phi_2}{\partial r_1^2} \leq 0$) then the enclosed region $\mathcal{R}\{\Phi_2\}$ is convex. The rate region is defined to be convex when a straight line connecting any two points inside the rate region is entirely enclosed in the rate region. In contrast, if the potential line is not concave, i.e., if it is convex or exhibits a non-stationary inflection point, then we describe its enclosed region as being concave; as in this situation, the aforementioned definition of a convex region does not hold. In summary, if Φ_i is concave, then $\mathcal{R}\{\Phi_i\}$ is convex, and $\mathcal{R}\{\Phi_i\}$ is concave otherwise.

Let $\Re(\cdot)$ be the real operation, the *inflection threshold* Q_1 is defined as:

$$Q_1 = \frac{\Re(\sqrt{(a - \alpha)(a - \alpha + acP_{\max})}) - \alpha}{ad}, \quad (21)$$

where Q_1 was derived such that $\text{sign}\left(\frac{\partial^2 \Phi_2}{\partial r_1^2}\right) = \text{sign}(P_1 - Q_1)$. Therefore, it suffices to study convexity or concavity of potential line Φ_2 by examining the sign of $(P_1 - Q_1)$. Note that the inflection threshold Q_1 only depends on system parameters a , b , c , d , and P_{\max} . The relation is nonlinear. By plugging in the respective values, the convexity behavior is assessed, and it can be decided whether time-sharing is needed.

Thus, Φ_2 can exhibit the following convexity behaviors:

- $Q_1 \geq P_{\max}$: then $P_1 - Q_1 \leq 0$ for all the range of P_1 , thus Φ_2 is concave. Operating via power control is optimal in leading the highest achievable data rate, and no time-sharing is needed. See Φ_2 in case (i) in Fig.6.
- $0 < Q_1 < P_{\max}$: Φ_2 exhibits a non-stationary inflection point when $P_1 = Q_1$ (see point D in Fig.6 and Fig.7). In this case:
 - for $0 < P_1 \leq Q_1$: line $\Phi(0 : Q_1, P_{\max})$ is concave, i.e., the potential line segment Φ_{AD} is concave as in Fig.6 case (ii) and Fig.7 (a). Operating via power control to trace the segment $\Phi(0 : Q_1, P_{\max})$ is optimal.

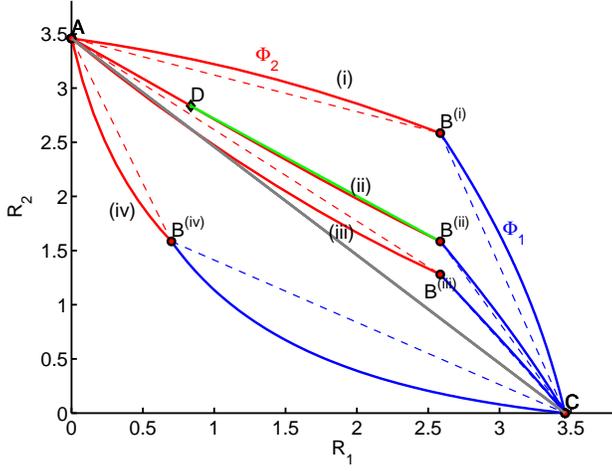


Fig. 6: $P_{\max} = 1$; (i): concave Φ_2 and Φ_1 ($[a, b; c, d] = [10, 1; 1, 10]$), (ii): Φ_2 with inflection point D, concave Φ_1 ($[10, 1; 4, 10]$), (iii): convex Φ_2 , concave Φ_1 ($[10, 1; 6, 10]$), (iv): convex Φ_2 and Φ_1 ($[10, 15; 4, 10]$)

- for $Q_1 \leq P_1 < P_{\max}$: line $\Phi(Q_1 : P_{\max}, P_{\max})$ is convex, i.e., the potential line segment $\Phi_{DB^{(ii)}}$ is convex as in Fig.6 case (ii) and Fig.7 (a). Therefore, operating via the time-sharing segment between the inflection point D and point B is optimal.
- $Q_1 \leq 0$: then $P_1 - Q_1 \geq 0$ for all the range of P_1 , thus Φ_2 is convex. See Φ_2 in cases (iii) and (iv) in Fig.6. Depending on the a, b, c, d , and P_{\max} parameters, it is optimal to apply time-sharing with the following options:
 - between point A and point B, see Fig.7 (c).
 - between point A and a point on the concave segment of Φ_1 , see Fig.7 (b).
 - between point A and point C, see Fig.7 (d). This is effectively Time Division Multiplexing (TDM), where each user transmits solely at any point of time. This form of dimension-orthogonality occurs when the interference is very strong rendering the cost too high for having simultaneous transmission. The upcoming subsection IV-B explores the optimality of operating via TDM, or equivalently, it explores when this cost is deemed too high.

By symmetry, the potential line Φ_1 exhibits similar convexity behavior: a) it is convex when $Q_2 \leq 0$, b) it is concave when $Q_2 \geq P_{\max}$, and c) it exhibits a non-stationary inflection point when $P_2 = Q_2$. Hereby, with $\beta = (b + bdP_{\max})$, the inflection threshold Q_2 is defined as:

$$Q_2 = \frac{\Re(\sqrt{(c - \beta)(c - \beta + acP_{\max})}) - \beta}{cb}. \quad (22)$$

Note that by virtue of how the inflection threshold is situated with respect to power P_i , whenever any of the potential lines exhibit an inflection point, the order of which segment is convex or concave is not arbitrary. It always starts concave in the segment closer to the coordinate axis. Explicitly, if Φ_2 and Φ_1 exhibit inflection points, then tracing the rate region

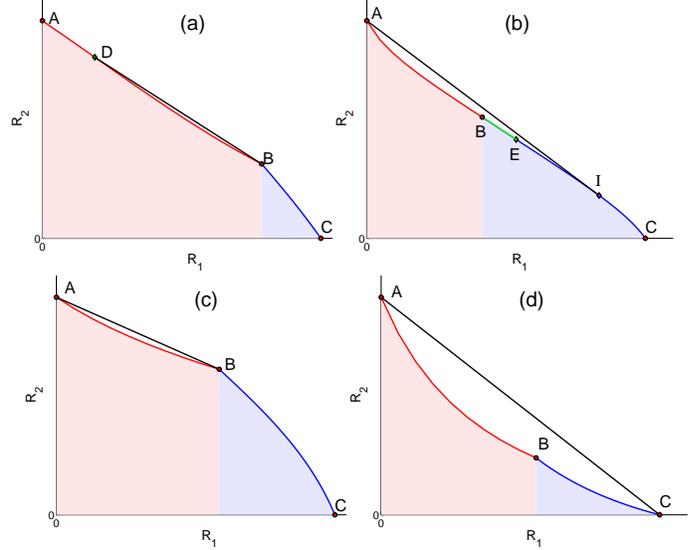


Fig. 7: types of time-sharing.

from left to right: Φ_2 is concave then transitions to convexity, and Φ_1 is convex then transitions to concavity.

B. When is TDM Optimal?

Discounting the case when Φ_1 or Φ_2 exhibit non-stationary inflection point for simplicity, and focusing on the case when both potential lines are convex (i.e., $Q_1 \leq 0$ and $Q_2 \leq 0$), it is important to know when TDM is optimal. Under the aforementioned assumption, this translates to determine when time-sharing between point A and point C is better than time-sharing through intermediate point B. This is done by comparing the y-axis ordinate of point B, $R_2(P_{\max}, P_{\max})$, relative to the y-axis ordinate from the straight line connecting points A and C at $r_1 = R_1(P_{\max}, P_{\max})$, denoted as $R_2^{TS}(r_1)|_{r_1=R_1(P_{\max}, P_{\max})}$, as shown in Fig.7 (d). Namely, TDM is optimal when $R_2^{TS}(r_1)|_{r_1=R_1(P_{\max}, P_{\max})} \geq R_2(P_{\max}, P_{\max})$; that is:

$$\frac{-\log_2(1+cP_{\max})}{\log_2(1+aP_{\max})} \log_2\left(1 + \frac{aP_{\max}}{1+bP_{\max}}\right) + \log_2(1+cP_{\max}) \geq \log_2\left(1 + \frac{cP_{\max}}{1+dP_{\max}}\right)$$

This leads to the following Lemma 2.

Lemma 2. *Operating via TDM (i.e. one transmitter solely transmitting at a certain time) is optimal in achieving the rate region when*

$$\frac{(1+cP_{\max})(1+dP_{\max})}{1+cP_{\max}+dP_{\max}} \geq \left(\frac{1+aP_{\max}+bP_{\max}}{1+bP_{\max}}\right)^\gamma \quad (23)$$

with $\gamma = \log_2(1+cP_{\max})/\log_2(1+aP_{\max})$.

Note that the condition found in Eq. (23) is a nonlinear relation between the interference channel variables of a, b, c, d , and P_{\max} . This motivates the following subsection IV-C to treat the 2-user symmetrical channel.

C. Symmetric 2-User Interference Channel

This subsection treats the symmetric 2-user channel, mainly analyzing the expression in Eq. (23) in order to derive clear

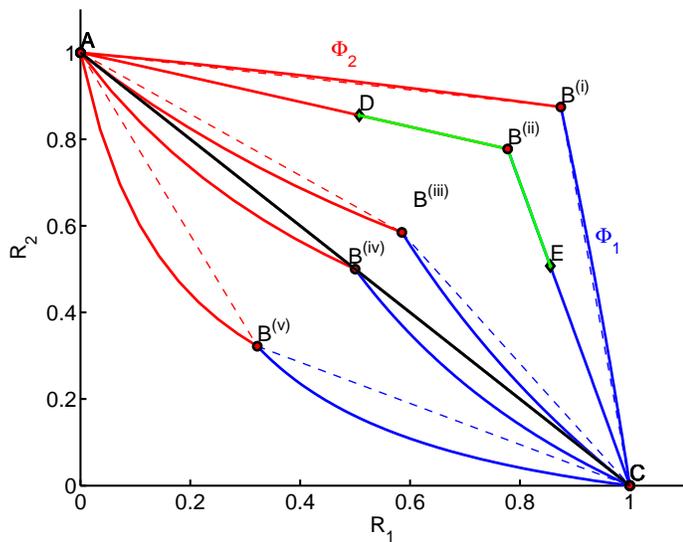


Fig. 8: symmetric 2–user rate region: $P_{\max} = 1$, $a = 1$, $b^{(i)} = 0.2$, $b^{(ii)} = 0.4$, $b^{(iii)} = 1$, $b^{(iv)} = \sqrt{2}$, and $b^{(v)} = 3$. The threshold b^* to switch to time-sharing is equal to $\sqrt{2}$ from Eq. (24).

insights. For the symmetric channel, $a = c$ and $b = d$, the expression in Eq. (23) simplifies, and leads to the following corollary on the TDM optimality condition:

Corollary 3. *For the symmetric 2–user interference channel, operating via TDM is optimal in achieving the rate region when*

$$b \geq \frac{\sqrt{1 + aP_{\max}}}{P_{\max}}. \quad (24)$$

In other words, when the interference is weak (i.e., b is below the threshold in Eq. (24)), then it is best for both transmitters to transmit at full power. When the interference increases and exceeds the threshold in Eq. (24), then the TDM scheme becomes optimal. In this scenario, the users can no longer share the same resource, and thus they have to use it in an orthogonal fashion. An example is illustrated in Fig.8. The application of the usage of the different types of time-sharing and the advantage over power control has been discussed in [27], when it was applied to a 2–sector interference channel in a cellular setting.

Remark: For high SNR (i.e. $aP_{\max} \gg 1$), Eq. (24) reduces to $bP_{\max} \geq \sqrt{aP_{\max}}$, which coincides with the results in [8]; where this condition marks the interference power threshold above which treating the interference as noise is no longer optimal in the Degrees of Freedom sense and no longer within a gap of 1-bit from achieving capacity.

In addition, Appendix B proves that the expression in Eq. (24) is a sufficient condition for both frontiers Φ_1 and Φ_2 to be convex, i.e., Q_1 and Q_2 are always ≤ 0 ; which is the starting necessary condition of subsection IV-B when treating the general asymmetric channel. Therefore, when b satisfies the expression in Eq. (24), the frontiers potential lines are always convex and TDM is optimal.

The extension of Corollary 3 to the n –user symmetric interference channel is provided in Appendix C, leading to

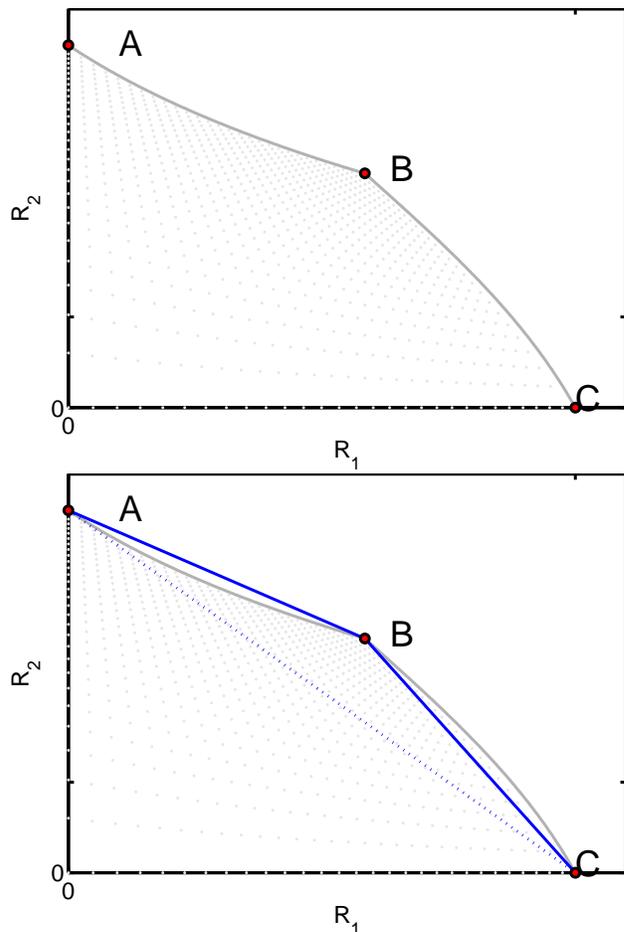


Fig. 9: 2–user crystallized rate region: (a) rate region through power control, (b) crystallized hull overlaid on top of the rate region

the following TDM optimality condition when:

$$b \geq \left(\frac{aP_{\max}}{(1 + aP_{\max})^{1/n} - 1} - 1 \right) \frac{1}{(n-1)P_{\max}}. \quad (25)$$

V. CRYSTALLIZED RATE REGION

Based on the discussion in the previous two sections, this section introduces a novel approach into simplifying the rate region in the n –dimensional space by having only an On/Off binary power control. This consequently leads to $2^n - 1$ corner points within the rate region. By forming a convex hull through time-sharing between those corner points, it thereby leads to what we denote a *crystallized* rate region [2]. The concept of the crystallized rate region has since been extended to the MIMO and the MIMO-OFDM channels [28], [29]. This section focuses on the crystallized rate region formulation and its evaluation.

As illustrated in Fig.9 for the 2–user case, the crystallized rate region is an approximation of the achievable rate region formed by a convex hull of straight lines connecting points A, B, and C. Those corner points are achieved through binary power control with the transmitters employing either zero or full power. For the 2–user interference channel, there exist 3

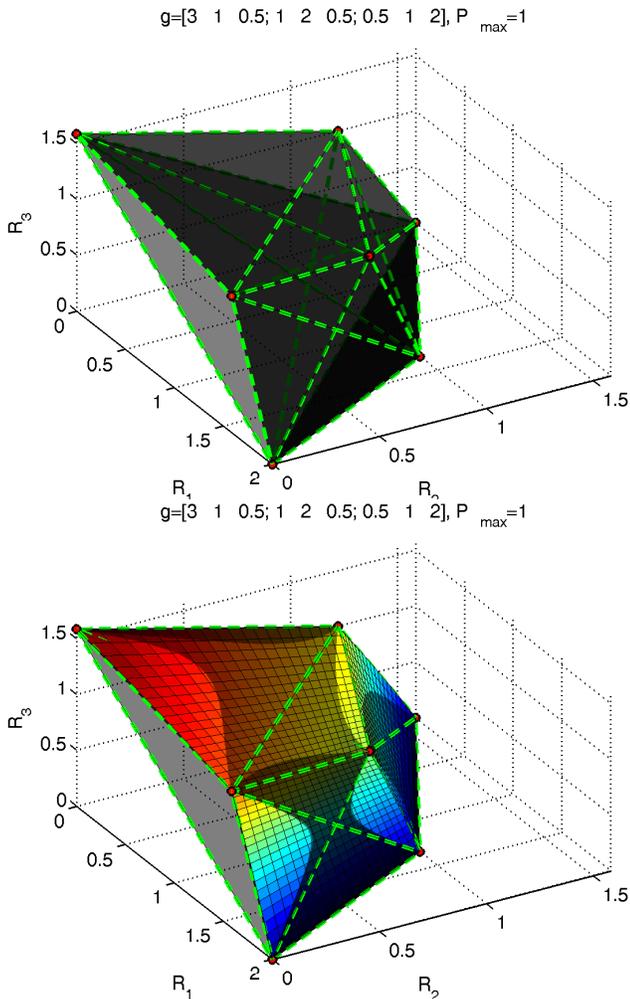


Fig. 10: 3-user crystallized rate region: (a) time-sharing crystallized hull, (b) crystallized hull overlaid on top of the power control rate region

corner points; similarly for the n -user case, there exist $2^n - 1$ corner points. Note that in Fig.9, the time-sharing convex hull connects point A and point C though point B; whereas if the interference is strong, the convex hull is formed by time-sharing A and C only. Fig.10 shows an example of the 3-user case where Fig.10 (a) is the crystallized hull, and Fig.10 (b) is the crystallized hull overlaid on top of the power control rate region. In the 2-user dimension, the time-sharing convex hull is a set of straight lines connecting two points. In the 3-user dimension, it is a set of polygon surfaces connecting three points, see Fig. 11. In the n -user dimension, each polygon is formed by connecting n points, and hence the number of such polygons is $\binom{2^n-1}{n}$.

A. System Time-sharing Coefficients and New Rates Equations

Recall that with power control, the achievable rates for the 2-user interference channel are given in Eq. (2). The paradigm with the crystallized rate region approach is different. Instead of formulating the problem as a power control problem for finding P_i , the crystallized rate region formulation focuses on finding the appropriate time-sharing coefficients of the

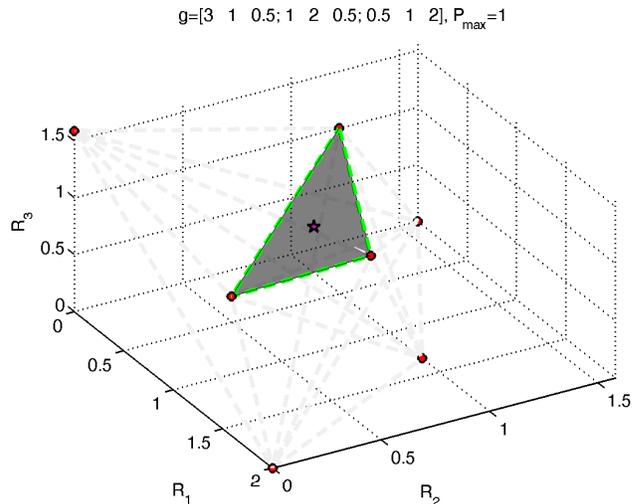


Fig. 11: a desired operating point in a time-sharing polygon

$2^n - 1$ corner points. For the 2-user case, let $\theta = [\theta_1, \theta_2, \theta_3]^T$, $\sum_i \theta_i = 1$, denote the *system* time-sharing coefficients vector of respective corner points $\Phi(P_{\max}, 0)$ (user 1 transmitting solely with a time-sharing coefficient θ_1), $\Phi(0, P_{\max})$ (user 2 transmitting solely with a time-sharing coefficient θ_2), and $\Phi(P_{\max}, P_{\max})$ (both users transmitting with a time-sharing coefficient θ_3). The reason for labeling θ a *system* time-sharing coefficients vector is to emphasize the combinatorial element in constructing the corner points, where the cardinality of $|\theta| = 2^n - 1$.

For 2-user case, in contrast with Eq. (2), the rates R_1 and R_2 for the crystallized region are:

$$\begin{aligned} R_1(\theta) &= \theta_1 \log_2(1 + aP_{\max}) + \theta_3 \log_2 \left(1 + \frac{aP_{\max}}{1 + bP_{\max}} \right), \\ R_2(\theta) &= \theta_2 \log_2(1 + cP_{\max}) + \theta_3 \log_2 \left(1 + \frac{cP_{\max}}{1 + dP_{\max}} \right). \end{aligned} \quad (26)$$

Any solution point on the crystallized hull lies somewhere on a time-sharing line connecting two points for the 2-user case; and similarly in the 3-user case, the solution point lies somewhere on a time-sharing plane connecting three points (see Fig.11), then by induction we obtain the following corollary:

Corollary 4. *For any solution point on the n -user crystallized rate region, the system time-sharing vector θ has at maximum n nonzero coefficients out of its $2^n - 1$ elements.*

Corollary 4 could also have been reached invoking the Carathéodory theorem [30], where, in the paper's context, any point lying on the hyper-surface of dimension $n - 1$ is enclosed in a convex set of n or fewer points inside the hyper-surface. Therefore, if those points are the corner points, any solution point on the $n - 1$ -dimensional hyper-surface will be the result of time-sharing at maximum n corner points.

Let $\alpha^{(k)}$ denote the transmitters power mask that characterizes the k^{th} corner point, specifically:

$$\begin{aligned} \alpha^{(k)} &= [\alpha_1^{(k)}, \dots, \alpha_i^{(k)}, \dots, \alpha_n^{(k)}]^T, \alpha_i^{(k)} \in \{0, 1\} \\ & i = 1, \dots, n, \text{ where } i \text{ is the transmitter index,} \\ & k = 1, \dots, 2^n - 1, \text{ where } k \text{ is the corner point index.} \end{aligned} \quad (27)$$

$\alpha_i^{(k)}$ is the action taken by the i^{th} user in characterizing the k^{th} corner point; the binary transmit action is either being silent or transmitting at full power. Let $\alpha_{-i}^{(k)}$ indicate the interferers transmit power mask, which is derived from vector $\alpha^{(k)}$ by excluding the i^{th} transmitter's action. The generalization for the n -user crystallized rates equations now follows as:

$$R_i(\theta) = \sum_{k=1}^{2^n-1} \theta_k \log_2 \left(1 + \frac{\alpha_i^{(k)} g_{ii} P_{\max}}{1 + \alpha_{-i}^{(k)T} \mathbf{g}_i P_{\max}} \right), \quad (28)$$

where g_{ii} is the desired channel gain from the i^{th} transmitter to the i^{th} receiver normalized by the noise power; \mathbf{g}_i is the noise-normalized channel gains vector received at the i^{th} receiver from all the transmitters excluding the i^{th} transmitter. The length of each vector $\alpha_{-i}^{(k)}$ and \mathbf{g}_i is $n-1$.

B. Evaluation of Crystallization

In this section, we compare the crystallized rate region and the rate region bounded by the power control potential lines. In effect, we are evaluating how much gain or loss results from completely replacing the traditional power control scheme (see Eq. (2)) with the time-sharing scheme between the corner points (see Eq. (26)). For this purpose, we consider the symmetric channel, where $a = 1$, $P_{\max} = 1$, and we increase the interference b to vary the signal to interference ratio $\text{SIR} = a/b$. Two metrics are used as a measure: (a) the area of the rate region, and (b) the maximum bit rate gap between the traditional power control and the crystallized rate region scheme.

The values of the area bounded by the power control potential lines and the area bounded by the crystallized rate region are plotted in Fig.12(a). For weak interference, or equivalently noise-limited regime, point B is used in constructing the crystallized region. As the interference increases beyond a certain threshold level, time-sharing through point B becomes suboptimal, and time-sharing A-C becomes optimal. The exact switching point between power control to time-sharing is given in Eq.(24). In Fig.12(a), this happens at the intersection of the time-sharing line through point B and the A-C time-sharing line. As indicated in Fig.12(a), there is no significant loss in the rate region area if time-sharing is used universally instead of traditional power control; in fact in some cases time-sharing offers considerable gain. Specifically, whenever the potential lines exhibit concavity, time-sharing loses to power control; whenever the potential lines exhibit convexity, time-sharing wins over power control. Different values of a also lead to the same conclusion.

The second measure, the maximum rate gap percentage between the rate achieved by traditional power control and the rate achieved by using the crystallized rate region is plotted in Fig.12(b). The maximum bit rate loss of using the crystallized hull compared to power control does not exceed 1%, and the crystallized strategy is therefore quite attractive. It is arguable that as the interference becomes the network bottleneck there is little reason to implement non-binary power control, and that dimension orthogonalization becomes the primary objective.

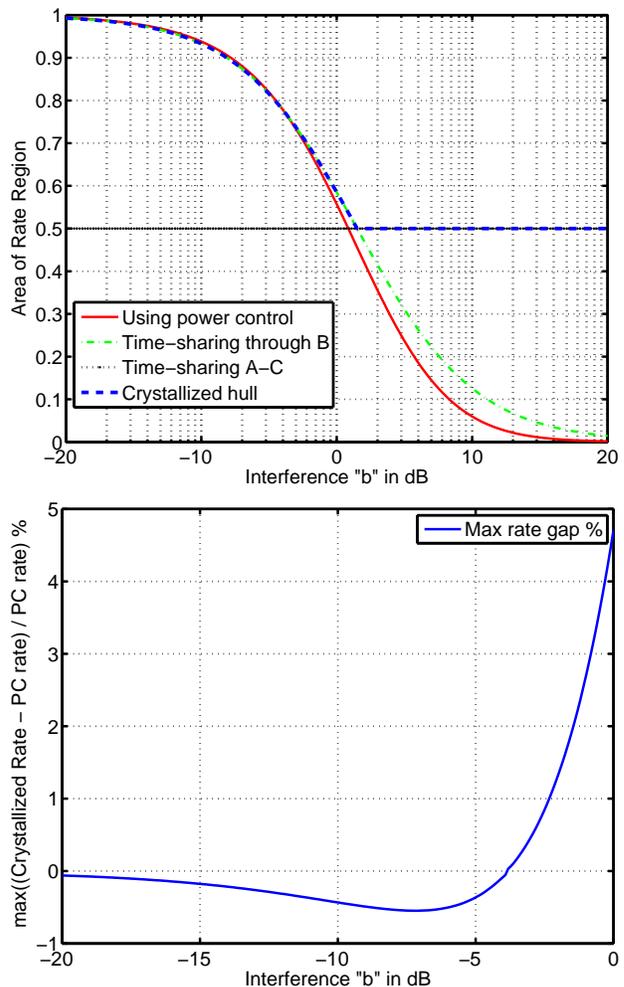


Fig. 12: (a) comparison of the areas bounded by crystallized hull and power control, (b) maximum rate gap percentage of the difference between crystallized hull and power control rates, with “b” from -20dB to 0dB

VI. CONCLUSION

The achievable rate region for the n -user interference channel was presented when the interference is treated as noise. The results were first found for the 2-user interference channel, then they were extended to the 3-user case to show the effect of adding an additional dimension. Subsequently, they were generalized for the n -user case. The n -user rate region was found to be the convex hull of the union of the n rate regions, where each rate region is upperbounded by a $(n-1)$ -dimensional hyper-surface characterized by having one of the transmitters transmitting at full power. For the 2-user interference channel, those hyper-surfaces become what the paper refers to as power control potential lines. Hence, to determine when time-sharing should be used in forming the achievable 2-user rate region, the paper studied the potential lines convexity behavior. Finally, the novel concept of the crystallized rate region was introduced and evaluated. The crystallized rate region is described by composing a time-sharing convex hull onto the $2^n - 1$ corner points obtained from On/Off binary power control. The evaluation of the crystallized rate region shows little value in the implementation

of non-binary power control compared to the use of binary power control in conjunction with time-sharing. In effect, the crystallized rate region approach offers a new perspective of looking at the achievable rate region of the interference channel.

APPENDIX A

PROOF THAT $R_2(P_2)$ IS MONOTONICALLY INCREASING IN P_2

Proof: Effectively, Eq. (5) is in the form of $f(1 + g(x))$. As $f(\cdot)$ is monotonically increasing in its argument, it suffices to prove that $g(x)$ is monotonically increasing in x . Therefore, define $g(P_2)$ as:

$$g(P_2) = \frac{acP_2}{a + d(1 + bP_2)(2^{r_1} - 1)}, \quad (29)$$

$$\frac{\partial g(P_2)}{\partial P_2} = \frac{ac(a + d(2^{r_1} - 1))}{(a + d(1 + bP_2)(2^{r_1} - 1))^2}. \quad (30)$$

The numerator in Eq. (30) is nonzero if $a \neq 0$ and $c \neq 0$ ($a = 0$ or $c = 0$ are the trivial cases where the rate region is either a line or the point zero). As $r_1 \geq 0$, then $(2^{r_1} - 1) \geq 0$. Thus $\partial g(P_2)/\partial P_2$ is always > 0 for non-trivial cases of a and c . Thus, $g(P_2)$ is monotonically increasing in P_2 , and equivalently $R_2(P_2)$ is monotonically increasing in P_2 . ■

APPENDIX B

PROOF THAT EQ. (24) IS A SUFFICIENT CONDITION FOR BOTH Φ_1 AND Φ_2 TO BE CONVEX

Proof: For the symmetric case, $Q_1 = Q_2 = Q_{sym}$ can be written as

$$Q_{sym} = \frac{\Re(\sqrt{(a - \theta)(a - \theta + a^2 P_{\max})}) - \theta}{ab},$$

where $\theta = b + b^2 P_{\max}$. Q_{sym} can also be written in this form:

$$Q_{sym} = \frac{\Re(\sqrt{T_1 T_2}) - \theta}{ab},$$

where $T_1 = a - \theta = a - b - b^2 P_{\max}$, and $T_2 = a - \theta + a^2 P_{\max}$. From the expression in Eq. (24), a can be alternatively upper-bounded as $a \leq (b^2 P_{\max} - 1/P_{\max})$. Therefore, T_1 is upper-bounded as:

$$T_1 \leq -1/P_{\max} - b. \quad (31)$$

From Eq. (31), T_1 is always negative. T_2 however can be positive or negative, evaluated as follows:

- $T_2 \geq 0$: then $\Re(\sqrt{T_1 T_2}) = 0$, and as θ is always positive, then $Q_{sym} \leq 0$.
- $T_2 \leq 0$: $\Re(\sqrt{T_1 T_2})$ is ≥ 0 . In this case, the numerator of Q_{sym} can be written as:

$$\text{num}(Q_{sym}) = \sqrt{(\theta - a)(\theta - a - a^2 P_{\max})} - \theta.$$

Given the fact that $(\theta - a - a^2 P_{\max}) \leq (\theta - a)$, then $\text{num}(Q_{sym})$ can be upper-bounded as:

$$\text{num}(Q_{sym}) \leq \sqrt{(\theta - a)^2} - \theta \leq -a \leq 0.$$

Hence, the frontiers potential lines Φ_1 and Φ_2 are convex. ■

APPENDIX C

EXTENSION OF COROLLARY 3 TO THE n -USER CASE

Focusing on the (R_1, R_2) 2D rate region of an n -user symmetrical rate region, if Eq. (24) is satisfied, Appendix B proved that it is a sufficient condition to make sure that Φ_1 and Φ_2 are convex and that they do not have an inflection point. Let b_2^* denote the threshold value in Eq. (24). Hence, if the interference channel gain $b \geq b_2^*$, then Φ_1 and Φ_2 in the (R_1, R_2) 2D rate region are convex. As more users transmit, they will cause additional interference power on R_1 and R_2 . Taking the 3-user case as an example, if the 3^{rd} user transmits, then $R_1(P_{\max}, P_2, P_3) = \log_2(1 + \frac{aP_{\max}}{1 + bP_2 + bP_3})$. Projecting the 3D rate region into the (R_1, R_2) 2D plane will lead to $R_1(P_{\max}, P_2) = \log_2(1 + \frac{aP_{\max}}{1 + b'P_2})$, where $b' = b + bP_3/P_2$, which is greater than b . Thus, if the (R_1, R_2) 2D rate region have a convex Φ_1 and Φ_2 , then $b \geq b_2^*$; and projecting higher n -dimensional rate region frontiers into the (R_1, R_2) 2D plane will always result in power control frontiers, denoted as Φ'_1 and Φ'_2 , that are always convex – due to the fact that they can be obtained using an interference power gain b' where $b' \geq b \geq b_2^*$.

The goal of this appendix section is to find the n -user symmetrical interference channel gain threshold, denoted as b_n^* , such that when $b \geq b_n^*$, TDM is optimal in leading higher achievable rate region. Thus, as a starting assumption (which is proved to hold later on), assume $b_n^* \geq b_2^*, \forall n \geq 2$; which therefore leads to the property that all the n -dimensional hypersurfaces are convex – as the projections on all the pairwise 2D planes result in convex power control frontiers.

With a TDM solution, users transmissions are orthogonal in time, and therefore, whenever user i transmits all other users are silent. The maximum TDM rate achievable for user i is when $P_i = P_{\max}$, and in the context of an n -user symmetrical channel, it is equal to $R_i = R_S^{TDM} = \log_2(1 + aP_{\max})$, and $R_j = 0, \forall j \neq i$. Therefore, whenever user i transmits under TDM, its maximum achievable rate in the n -dimensional rate region is a point on the i^{th} axis with a value equals to R_S^{TDM} ; let Φ_i^{TDM} denotes such a point.

Let \mathcal{H} be a hyperplane formed by connecting via time-sharing the $\Phi_i^{TDM}, i = 1, \dots, n$ points. Define the origin point O as the point with coordinates $R_i^O = 0, \forall i$. Define point B on the power control achievable rate region when all users transmits at the same time; thus point B coordinates are $R_i^B = \log_2\left(1 + \frac{aP_{\max}}{1 + (n-1)bP_{\max}}\right), \forall i$. Define point B' on the time-sharing hyperplane \mathcal{H} when all users transmits in TDM for an equal amount of time. Let \mathbf{OB} denotes a vector from point O to point B, and similarly, \mathbf{OB}' denotes a vector from point O to point B'. Let $\|\cdot\|$ denotes the vector length; for instance, for \mathbf{OB} , $\|\mathbf{OB}\|$ is the distance from point O to point B. Therefore, a time-sharing TDM solution is optimal when $\|\mathbf{OB}'\| \geq \|\mathbf{OB}\|$; which has the interpretation of the hyperplane \mathcal{H} leading higher achievable rate region than the power control hyper-surfaces frontiers.

Based on the aforementioned coordinates of point B, $\|\mathbf{OB}\| = \sqrt{n} \log_2(1 + \frac{aP_{\max}}{1 + (n-1)bP_{\max}})$. For $\|\mathbf{OB}'\|$, it is the shortest distance from point O to the hyperplane \mathcal{H} ; which is equal to the absolute value of the dot product of the unit normal

vector of \mathcal{H} and the vector \mathbf{OK} , where point K is a point on \mathcal{H} . The unit normal vector of \mathcal{H} is equal to $\frac{1}{\sqrt{n}}[1, \dots, 1]^T$. By choosing point K to be equal to the point Φ_1^{TDM} on the R_1 axis, $\|\mathbf{O}\Phi_1^{TDM}\| = [R_S^{TDM}, 0, \dots, 0]^T$, where R_S^{TDM} has been defined earlier to be equal to $\log_2(1 + aP_{\max})$. Therefore, $\|\mathbf{OB}'\| = \frac{1}{\sqrt{n}} \log_2(1 + aP_{\max})$.

Hence, a time-sharing TDM solution is optimal when

$$\frac{1}{\sqrt{n}} \log_2(1 + aP_{\max}) \geq \sqrt{n} \log_2 \left(1 + \frac{aP_{\max}}{1 + (n-1)bP_{\max}} \right). \quad (32)$$

Expanding Eq. (32) leads to Eq. (25). The threshold $b_n^* = \left(\frac{aP_{\max}}{(1+aP_{\max})^{1/n} - 1} - 1 \right) \frac{1}{(n-1)P_{\max}}$ is monotonically increasing for a positive n , which starts at Eq. (24) for $n = 2$ and flattens out asymptotically for large n . Using the first order approximation of $(1 + aP_{\max})^{1/n}$ to $1 + \frac{1}{n} \ln(1 + aP_{\max})$, b_n^* asymptotically converge to:

$$\lim_{n \rightarrow \infty} b_n^* = a / \ln(1 + aP_{\max}),$$

independent of the number of users n .

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