Tight Upper Bound on the Outage Probability of QSTBC

Aydin Sezgin, *Member, IEEE*, and Eduard A. Jorswieck, *Member, IEEE*

***Abstract*— Recently, the statistical properties of the equivalent channel representation of a multiple-input-multiple output (MIMO) system employing rate one quasi-orthogonal space-time block codes (QSTBC) for $n_T = 2^n$, $n \geq 2$ transmit antennas was characterized. It was shown, that these QSTBC were capable of achieving a significant fraction of the outage mutual information of a MIMO system. In this letter, based on the Minkowski's determinant inequality and the Meijer's G-function we derive an upper bound for the fraction of outage probability achieved with QSTBC for the general case of $n_T = 2^n$,$n \geq 2$ transmit antennas. Simulations results show that this bound gets tight for all signal-to-noise-ratios (SNR) values by increasing the number of receive antennas.**

***Index Terms*— Space-time coding, outage, Meijer's G-function, performance bound.**

I. Introduction

THERE has been considerable work on a variety of schemes which exploit multiple antennas at both the transmitter and receiver in order to obtain transmit and receive diversity and therefore increase the reliability of the system, e.g., orthogonal space-time block codes (OSTBC) [1], [2].

The performance of OSTBC with respect to mutual information has been analyzed (among others) in [3]–[5]. Unfortunately, the Alamouti space-time code for two transmit and one receive antennas is the only OSTBC, which achieves the capacity. Therefore, [6]–[8] designed quasi-orthogonal space-time block code (QSTBC) with transmission rate one for $n_T = 4$ transmit antennas and later on [9]–[12] generalized to $n_T = 2^n$ transmit antennas . The performance of QSTBC with respect to outage mutual information (OMI) for the case of four transmit and one receive antennas has been analyzed via simulations in [7] and it was shown, that the QSTBC are capable to achieve a significant portion of the OMI. An upper bound on the outage performance of QSTBC for special case of $n_T = 4$ was derived in [11], [12] involving an infinite sum, which is, however, not tight. In this work, we derive an upper bound on the outage performance of QSTBC for the general case of $n_T = 2^n$, $n \geq 2$ transmit antennas, which is tight for all SNR values, especially for more than one receive antenna.

In [13], a differential modulation scheme based on QSTBC was proposed, whereas the combination of precoding techniques and QSTBC was discussed in [14], [15]. The analysis of QSTBC for frequency-selective channels is given in [16].

The remainder of this letter is organized as follows. In Section II, we introduce the system model and establish the notation. The design of QSTBC for 2^n transmit antennas is shown in section III. The analysis of the outage probability achieved with QSTBC is done in Section IV, followed by some simulations and concluding remarks in Section V and VI.

II. System Model

We consider a system with n_T transmit and n_R receive antennas. Our system model is defined by

$$\mathbf{Y} = \mathbf{G}_{n_T}\mathbf{H} + \mathbf{N}\,, \tag{1}$$

where $\mathbf{G}_{n_T}$ is the $(T \times n_T)$ transmit matrix, $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_{n_R}]$ is the $(T \times n_R)$ receive matrix, $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_{n_R}]$ is the $(n_T \times n_R)$ channel matrix, and $\mathbf{N} = [\mathbf{n}_1, \ldots, \mathbf{n}_{n_R}]$ is the complex $(T\times n_R)$ additive white Gaussian noise (AWGN) matrix, where an entry $\{n_{ti}\}$ of $\mathbf{N}$ $(1 \leq i \leq n_R)$ denotes the complex noise at the ith receiver for a given time $t\,(1 \leq t \leq T)$. The real and imaginary parts of n_{ti} are independent and $\mathcal{N}(0,n_T/(2\rho))$ distributed, where ρ the signal-to-noise ratio (SNR). An entry of the channel matrix is denoted by $\{h_{ji}\}$. This represents the complex gain of the channel between the jth transmit $(1 \leq j \leq n_T)$ and the ith receive $(1 \leq i \leq n_R)$ antenna, where the real and imaginary parts of the channel gains are independent and normal distributed random variables with $\mathcal{N}$(0,1/2) per dimension. The channel matrix is assumed to be constant for a block of T symbols and changes independently from block to block. The average power of the symbols transmitted from each antenna is normalized to be one, so that the average power of the received signal at each receive antenna is n_T. It is further assumed that the transmitter has no channel state information (CSI) and the receiver has perfect CSI.

III. Code Construction

A space time block code is defined by its transmit matrix $\mathbf{G}_{n_T}$ with entries $\{x_j\}_{j=1}^{p}$, which are elements of the vector $\mathbf{x} = [x_1, \ldots, x_p]^T$. The rate R of a space-time code is defined as $R = p/T$. In this paper, we focus on rate one QSTBC with $n_T = T$. It follows that $p = n_T$.

Starting with the well known Alamouti scheme [1] for $n_T = 2$ transmit antennas

$$\mathbf{G}_2(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ x_2^* & -x_1^* \end{bmatrix}\,,$$

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The authors were with the Fraunhofer-Institute for Telecommunications, Heinrich-Hertz-Institut, Einsteinufer 37, D-10587 Berlin, Germany.

A. Sezgin is now with the Information Systems Laboratory, Stanford University, Stanford, CA 94305-9510 USA (e-mail: sezgin@stanford.edu).

E. Jorswieck is now with the Royal Institute of Technology, 10044 Stockholm, Sweden (e-mail: Eduard.Jorswieck@ee.kth.se).

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the transmit matrix of the QSTBC with $n_T = 2^n$ ($n \geq 2$) is constructed in the following way

$$\mathbf{G}_{n_T}\left(\{x_j\}_{j=1}^{n_T}\right) = \left[\begin{array}{cc} \mathbf{G}_{\frac{n_T}{2}}\left(\{x_j\}_{j=1}^{\frac{n_T}{2}}\right) & \mathbf{G}_{\frac{n_T}{2}}\left(\{x_j\}_{j=\frac{n_T}{2}+1}^{n_T}\right) \\ \mathbf{G}_{\frac{n_T}{2}}\left(\{x_j\}_{j=\frac{n_T}{2}+1}^{n_T}\right)\mathbf{\Theta} & -\mathbf{G}_{\frac{n_T}{2}}\left(\{x_j\}_{j=1}^{\frac{n_T}{2}}\right)\mathbf{\Theta} \end{array}\right],$$

where $\mathbf{\Theta}$ is given by $\mathbf{\Theta} = \mathrm{diag}\left(\{(-1)^{j-1}\}_{j=1}^{\frac{n_T}{2}}\right)$.

After some manipulations, particularly channel matched filtering [11], [12], the system model in (1) can be decomposed into two equivalent system models for $\mathbf{x}_\mathrm{o}$ and for $\mathbf{x}_\mathrm{e}$, where $\mathbf{x}_\mathrm{o} = [x_1, x_3, \ldots, x_{n_T-1}]$ and $\mathbf{x}_\mathrm{e} = [x_{n_T}, x_{n_T-2}, \ldots, x_2]$. The equivalent system model for $\mathbf{x}_\mathrm{o}$ (and similarly for $\mathbf{x}_\mathrm{e}$) is given as

$$\mathbf{y}_\mathrm{o} = \widetilde{\mathbf{H}}\mathbf{x}_\mathrm{o} + \widetilde{\mathbf{n}}\,. \tag{2}$$

For illustration, we present two examples for the case of $n_T = 4$ and $n_T = 8$ transmit antennas.

Example 3.1: ($n_T = 4$ transmit antennas) For the case of $n_T = 4$ transmit antennas we have

$$\mathbf{G}_4(\{x_j\}_{j=1}^4) = \left[\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ x_2^* & -x_1^* & x_4^* & -x_3^* \\ x_3 & -x_4 & -x_1 & x_2 \\ x_4^* & x_3^* & -x_2^* & -x_1^* \end{array}\right].$$

In this case, the decomposed system model for $\mathbf{x}_\mathrm{o}$ (and similarly for $\mathbf{x}_\mathrm{e}$, cf. (2)) can be written as

$$\mathbf{y}_\mathrm{o} = \widetilde{\mathbf{H}}\left[\begin{array}{c} x_1 \\ x_3 \end{array}\right] + \widetilde{\mathbf{n}}\,,$$

with a non-orthogonal

$$\widetilde{\mathbf{H}} = \left[\begin{array}{cc} \lambda & i\alpha \\ -i\alpha & \lambda \end{array}\right],$$

where λ and α are given as

$$\lambda = \sum_{i=1}^{n_R}\sum_{j=1}^{n_T}|h_{ji}|^2 \text{ and}$$
$$\alpha = \sum_{i=1}^{n_R} 2\mathrm{Im}(h_{1i}^* h_{3i} + h_{4i}^* h_{2i}),$$

respectively.

Example 3.2: ($n_T = 8$ transmit antennas) Here, $\widetilde{\mathbf{H}}$ is given as

$$\widetilde{\mathbf{H}} = \left[\begin{array}{cccc} \lambda & i\alpha & i\beta & \delta \\ -i\alpha & \lambda & -\delta & i\beta \\ -i\beta & -\delta & \lambda & i\alpha \\ \delta & -i\beta & -i\alpha & \lambda \end{array}\right],$$

where

$$\lambda = \sum_{i=1}^{n_R}\sum_{j=1}^{n_T}|h_{ji}|^2,$$
$$\alpha = \sum_{i=1}^{n_R} 2\mathrm{Im}(h_{1i}^* h_{3i} + h_{4i}^* h_{2i} + h_{5i}^* h_{7i} + h_{8i}^* h_{6i}),$$
$$\beta = \sum_{i=1}^{n_R} 2\mathrm{Im}(h_{1i}^* h_{5i} + h_{6i}^* h_{2i} + h_{3i}^* h_{7i} + h_{8i}^* h_{4i}), \text{ and}$$
$$\delta = \sum_{i=1}^{n_R} 2\mathrm{Re}(h_{1i}^* h_{7i} + h_{8i}^* h_{2i} - h_{3i}^* h_{5i} - h_{6i}^* h_{4i}).$$

Since $\widetilde{\mathbf{n}}$ in (2) is colored noise, the next step is to perform pre-whitening. To compute the pre-whitening filter, we need the eigenvalue decomposition (EVD) of $\widetilde{\mathbf{H}}$, which is given as $\widetilde{\mathbf{H}} = \mathbf{V}\mathbf{S}\mathbf{V}^H$, where $\mathbf{V}$ is a unitary matrix and $\mathbf{S}$ is a diagonal matrix with eigenvalues on its diagonal. Therefore, the pre-whitening filter is given as $\mathbf{F}_\mathrm{PW} = \mathbf{S}^{-1/2}\mathbf{V}^H$. By multiplying $\mathbf{F}_\mathrm{PW}$ to (2) from left we arrive at

$$\widehat{\mathbf{y}}_\mathrm{o} = \widehat{\mathbf{H}}\mathbf{x}_\mathrm{o} + \mathbf{w}\,, \tag{3}$$

where the entries of $\mathbf{w}$ are mutually i.i.d. Gaussian random variables again.

Example 3.3: In the case of $n_T = 4$ transmit antennas, $\widehat{\mathbf{H}}$ in (3) is given as

$$\widehat{\mathbf{H}} = \left[\begin{array}{cc} \mu_1 & i\mu_1 \\ \mu_2 & -i\mu_2 \end{array}\right], \tag{4}$$

and $\mu_1 = \sqrt{\frac{\lambda+\alpha}{2}}$, $\mu_2 = \sqrt{\frac{\lambda-\alpha}{2}}$.

Let us now decompose $\widehat{\mathbf{H}}$ as

$$\widehat{\mathbf{H}} = \mathbf{S}^{1/2}\mathbf{U}^H\,, \tag{5}$$

where $\mathbf{S}^{1/2}$ contains the nonnegative singular values of $\widehat{\mathbf{H}}$ and $\mathbf{U}$ is a unitary matrix.

Lemma 3.1: For any n_T and n_R, the eigenvalues μ_i of $\frac{2}{n_T}\mathbf{S}$ are independent and each has a chi-square distribution with $4n_R$ degrees of freedom.

Proof: The proof is given in [11], [12].

IV. Outage Probability P_{out}

The mutual information of a MIMO system with n_T transmit and n_R receive antennas is given as [17]

$$I = \log_2 \det\left(\mathbf{I}_{n_R} + \frac{\rho}{n_T}\mathbf{H}\mathbf{H}^H\right).$$

The portion of the mutual information achieved with QSTBC is

$$\begin{aligned} I_Q &= \frac{2}{n_T}\log_2 \det\left(\mathbf{I}_{n_T/2} + \frac{\rho}{n_T}\mathbf{S}\right) \\ &= \frac{2}{n_T}\log_2 \prod_{i=1}^{n_T/2}\left(1 + \frac{\rho}{2}\mu_i\right). \end{aligned} \tag{6}$$

The outage probability P_{out} achievable with QSTBC is defined as the probability that I_Q is smaller than a certain rate R, i.e.

$$P_{out}(R, n_T, n_R, \rho) = \Pr[I_Q < R]\,.$$

Theorem 4.1: The outage probability of a QSTBC for an arbitrary number of receive and $n_T = 2^n$, $n \geq 2$ transmit antennas is upper bounded by

$$P_{out}(R, n_T, n_R, \rho) \leq \frac{\tilde{R}}{((2n_R-1)!)^{\frac{n_T}{2}}} \times G_{1\ \frac{n_T}{2}+1}^{\frac{n_T}{2}\ 1}\left(\tilde{R}\,\middle|\,\begin{array}{l} 0 \\ \mathbf{1}_{\frac{n_T}{2}}^T(2n_R-1), -1 \end{array}\right) \tag{7}$$

where $G_{pq}^{m\,n}(\cdot)$ is the Meijer's G-function defined in [18, p.1068, eq.9.301], $\mathbf{1}_{\frac{n_T}{2}}$ is the all ones vector of length $\frac{n_T}{2}$ and $\tilde{R} = \left(\left(2^R-1\right)\frac{2}{\rho}\right)^{\frac{n_T}{2}}$.

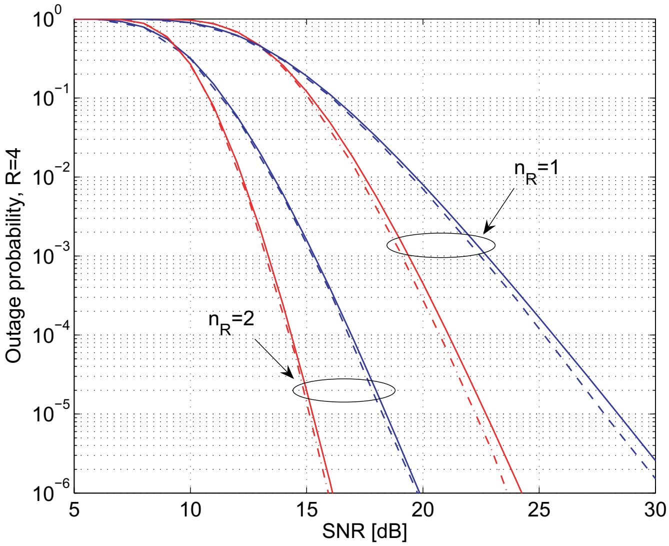

Fig. 1. Outage probabilities of QSTBC ($n_T = 4$,dashed lines and $n_T = 8$, dash-dotted lines) and upper bound (solid lines) with $n_R = 1, 2$ receive antennas and rate $R = 4$.

Proof: By applying Minkowski's determinant inequality [19, p.482]$(\det(\mathbf{I}_n + \mathbf{C}) \geq (1 + \det(\mathbf{C})^{\frac{1}{n}})^n)$ to (6), we arrive at

$$I_Q \geq \log_2 \left(1 + \frac{\rho}{n_T} \det(\mathbf{S})^{\frac{2}{n_T}}\right) , \quad (8)$$

The outage probability P_{out} achievable with QSTBC may then upper bounded by

$$\begin{aligned} P_{out}(R, n_T, n_R, \rho) =& \Pr[I_Q < R] \\ & \leq \Pr\left[\det(\mathbf{S}) < \left((2^R - 1)\frac{n_T}{\rho}\right)^{\frac{n_T}{2}}\right] . \end{aligned}$$

Using Lemma 3.1, the probability density function (pdf) of each μ_i is given as

$$p_{\mu_i}(\mu_i) = \frac{1}{\Gamma(2n_R)}(\mu_i)^{2n_R-1}e^{-\mu_i} \quad (9)$$

From [20], we can infer that the pdf of the product of i.i.d chi-square distributed variables $y = \prod_i^{\frac{n_T}{2}} \mu_i$ is given as

$$p_y(y) = \frac{1}{\Gamma(2n_R)^{\frac{n_T}{2}}} G^{\frac{n_T}{2}\,0}_{0\,\frac{n_T}{2}}\left(y \left| \begin{array}{l} [\,] \\ \mathbf{1}^T_{\frac{n_T}{2}}(2n_R - 1) \end{array}\right.\right). \quad (10)$$

Integrating over the pdf of y and using [18, p.897, eq.7.811.2] we arrive at the upper bound on the outage probability given as in (7). That concludes the proof. ■

V. Simulations

In Fig. 1, P_{out} of QSTBC with $n_T = 4$ and $n_T = 8$ transmit and $n_R = 1, 2$ receive antennas is depicted. From the Fig. we observe that the upper bound in (7) is tight for low to average SNR values in the case of one receive antenna. For $n_R = 2$ and more receive antennas (not depicted here), the bound is tight for all SNR values.

VI. Conclusion

In this letter, we derived an upper bound on the outage probability of QSTBC for 2^n transmit and an arbitrary number of receive antennas based on Minkowski's determinant inequality and Meijer's G-function. From simulation results, we observed that the upper bound is tight for low and average SNR values in case of one receive antenna and gets tight for all SNR values by increasing the number of receive antennas.

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