Impact of Spatial Correlation on the Performance of Orthogonal Space-Time Block Codes

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***Abstract*—We analyze the impact of transmitter and receiver spatial correlation on the performance of a multiple-input multiple-output (MIMO) system which applies an orthogonal space-time block code with no channel state information at the transmitter and perfect channel state information at the receiver. We derive a general formula for the bit error rate of a MIMO system with arbitrary number of transmit and receive antennas as a function of the correlation at the transmitter and the receiver. We prove that the diversity advantage is given by $M \cdot N$ if M is the rank of the transmit correlation matrix and N the rank of the receive correlation matrix, respectively.**

***Index Terms*—Bit error performance, correlation, multiple-input multiple-output (MIMO), space-time coding.**

I. Introduction

THE increasing need for fast and reliable wireless communication links has opened the discussion about systems with multiple antennas both located at the transmitter and the receiver, so called multiple-input multiple-output (MIMO) systems [1].

One concept for achieving high portions of the capacity and performance gains in MIMO systems is space-time coding [2], [3]. For the design of space-time codes it is assumed that the receiver has perfect channel state information (CSI) while the transmitter has no CSI. The bit error performance of the Alamouti scheme is studied in [4], the symbol error performance of OSTBC is analyzed in [5].

Most of the results regarding the ergodic and outage capacity and the bit error rates of single user MIMO systems assume that the transmit and receive antennas are uncorrelated. In reality, there can occur correlation at the transmit and the receive antenna array due to the placement of the array and the geometry in the transmission scenario. Especially at the base station, which is often un-obstructed, correlation between the antennas can occur. In this work, we compute the impact of transmit and receive correlation on the BER of OSTBC by analysis and simulation.

II. System and Channel Model

We consider a single-user MIMO system with n_T transmit and n_R receive antennas. The receiver has perfect CSI while the transmitter has no CSI. The transmitter applies an OSTBC. The noise at the receiver is complex iid distributed with variance σ_n^2. The total transmit power is constrained to P. We define the SNR as $\rho = P/\sigma_n^2$. The channel is assumed to be a flat fading channel with matrix entries $[h_i, j]_{i=1,j=1}^{n_T,n_R}$. The receiver applies a matched filter and we obtain at the output k of the matched filter at the receiver the following:

$$r_k = \left(\sqrt{\sum_{i=1}^{n_T}\sum_{j=1}^{n_R}|h_{i,j}|^2}\right) x_k + \tilde{n}_k \;\; \forall k = 1 \ldots n_T \qquad (1)$$

with $\tilde{n}k$ is complex iid with variance σ_n^2 because the normed matched filter matrix is unitary. The matched filter that leads to (1) is matched to an effective channel that takes into account the space-time code, not the actual physical channel.

The $n_R \times n_T$ channel matrix $\mathbf{H}$ for the case in which we have correlated transmit and correlated receive antennas is modeled as (with the same assumptions as in [6])

$$\mathbf{H} = \mathbf{R}_R^{\frac{1}{2}} \cdot \mathbf{W} \cdot \mathbf{R}_T^{\frac{1}{2}} \qquad (2)$$

with the $n_T \times n_T$ transmit correlation matrix $\mathbf{R}_T = \mathbf{U}_T\mathbf{D}_T\mathbf{U}_T^H$. $\mathbf{U}_T$ is the matrix with the eigenvectors of $\mathbf{R}_T$ and $\mathbf{D}_T$ is a diagonal matrix with the eigenvalues of the matrix $\mathbf{R}_T$. Similarly, the eigenvalue decomposition of the $n_R \times n_R$ receive correlation matrix is $\mathbf{R}_R = \mathbf{U}_R\mathbf{D}_R\mathbf{U}_R^H$. We denote the eigenvalues of the matrices $\mathbf{D}_R = \mathrm{diag}[\boldsymbol{\nu}]$ and $\mathbf{D}_T = \mathrm{diag}[\boldsymbol{\mu}]$ as $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_{n_T}]$ and $\boldsymbol{\nu} = [\nu_1, \ldots, \nu_{n_T}]$. To fix the average channel transmission power we assume that $\sum_{k=1}^{n_T}\mu_k = n_T$ and $\sum_{k=1}^{n_R}\nu_k = n_R$ and $\mu_i \geq 0$ and $\nu_i \geq 0$. The random matrix $\mathbf{W}$ has independent complex Gaussian identically distributed entries, i.e. $\mathbf{W} \sim \mathcal{CN}(0, \mathbf{I})$.

We assume that the entries in the channel matrix are complex Gaussian distributed. Therefore, we can describe the impact of the correlation at the transmitter and the receiver by considering the second moment, i.e. the covariance matrix. The analysis in [7] can not be applied in our scenario because the entries of the channel matrix in [7] are products of complex Gaussian distributed entries. It does not suffice to consider only the second moment in order to analyze the impact of correlation or keyholes. However, the authors in [7] argue that a MIMO system with 'many' keyholes converges to the common MIMO model by application of the central limit theorem.

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III. BER as a Function of Correlation

We follow the approach in [8] in order to derive the bit error rate (BER) of the OSTBC. Consider one output of the matched filter from (1), then the instantaneous SNR per bit is given by

$$\gamma = \left(\sum_{i=1}^{n_T} \sum_{j=1}^{n_R} |h_{i,j}|^2 \right) \rho. \tag{3}$$

The decision $\hat{x}_k = \operatorname{sign}(r_k)$ has bit error probability $\mathcal{Q}(\sqrt{2\gamma})$. Averaging the bit error probability over the probability density function (pdf) of the instantaneous SNR γ provides the BER [8]. The pdf of γ is a function of n_T and n_R and the correlation vectors $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Hence, the BER as a function of the SNR ρ, the number of transmit n_T and receive n_R antennas and the correlation $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ can be written as

$$\mathrm{BER} = \int_0^{\infty} \mathcal{Q}(\sqrt{2\gamma}) p(\gamma) d\gamma. \tag{4}$$

The probability density function of γ in (4) can be expressed in terms of its characteristic function $p(\gamma) = \int_{-\infty}^{\infty} \psi(\omega) e^{I \cdot \omega \cdot \gamma} d\omega$. I is defined as $I = \sqrt{-1}$. The characteristic function of the weighted sum of exponential distributed random variables in (3) is given by

$$\psi(\omega) = \prod_{i=1}^{n_T} \prod_{j=1}^{n_R} \frac{1}{1 + \rho I \cdot \mu_i \nu_i \omega}. \tag{5}$$

From (4) and (5) we have for the BER

$$\mathrm{BER}(\rho, n_T, n_R, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int_0^{\infty} \mathcal{Q}(\sqrt{2\gamma}) \int_{-\infty}^{\infty} \prod_{i=1}^{n_T} \prod_{j=1}^{n_R} \frac{1}{1 + \rho I \cdot \mu_i \nu_i \omega} e^{I \cdot \omega \cdot \gamma} d\omega d\gamma. \tag{6}$$

In general, the expression in (6) is difficult to analyze directly. We observe that transmitter and receiver correlation have the same impact on the error performance due to the symmetry in the pdf. The diversity advantage d is defined as the power of the SNR ρ in the denominator of (6) for the BER [2]. In the following, we study the diversity advantage d in order to analyze the behavior of the BER for high SNR. The following theorem characterizes the impact of correlation on the diversity advantage of OSTBC.

Theorem 1: Given the transmit correlation vector $\boldsymbol{\mu}$ and receive correlation vector $\boldsymbol{\nu}$ for a $n_T \times n_R$ MIMO systems applying an OSTBC. Denote the number of transmit and receive correlation eigenvalues which are greater than zero as M and N respectively. The diversity gain is given by

$$d = M \cdot N.$$

Remark: The Theorem 1 states that as long as the correlation matrices $\mathbf{R}_T$ and $\mathbf{R}_R$ have full rank, the full diversity advantage of $n_T \cdot n_R$ is achieved. This characterizes the robustness of OSTBC against correlation.

Proof: The transmit and receeive correlation eigenvalues are sorted in decreasing order. By assumption the last $n_T - M$ transmit correlation eigenvalues are zero, i.e. $\mu_{M+1} = \mu_{M+2} = \ldots = \mu_{n_T} = 0$ and the last $n_R - N$ receive correlation eigenvalues are zero, i.e. $\nu_{N+1} = \nu_{N+2} = \ldots = \nu_{n_R}$. We obtain for the BER in (6)

$$\mathrm{BER}(\rho, n_T, n_R, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int_0^{\infty} \mathcal{Q}(\sqrt{2\gamma}) \int_{-\infty}^{\infty} \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{1}{1 + \rho I \cdot \mu_i \nu_i \omega} e^{I \cdot \omega \cdot \gamma} d\omega d\gamma. \tag{7}$$

We can upper bound the BER in (7) by

$$\mathrm{BER}(\rho, n_T, n_R, \boldsymbol{\mu}, \boldsymbol{\nu}) \leq \int_0^{\infty} \mathcal{Q}(\sqrt{2\gamma}) \times \left(\prod_{j=1}^{N} \prod_{i=1}^{M} \underbrace{\int_{-\infty}^{\infty} \left(\frac{1}{1 + \rho I \cdot \mu_i \nu_i \omega} e^{I \cdot \omega \cdot \gamma} \right)^{M \cdot N} d\omega}_{\phi(\gamma)} \right)^{1/(M \cdot N)} d\gamma. \tag{8}$$

The inner integral $\phi(\gamma)$ in (8) is given by

$$\phi(\gamma)^{\frac{1}{M \cdot N}} = \frac{M \cdot N}{(\Gamma(M \cdot N + 1))^{\frac{1}{MN}}} \frac{\gamma}{(\gamma)^{\frac{1}{MN}}} \frac{e^{-\frac{\gamma}{\rho \mu_i \nu_j}}}{\rho \mu_i \nu_j} \tag{9}$$

with the Gamma function $\Gamma(K)$. We obtain with (9) in (8) and with $e^{-x} \leq 1$ for $x \geq 0$ the upper bound on the BER

$$\begin{aligned} &\mathrm{BER}(\rho, n_T, n_R, \boldsymbol{\mu}, \boldsymbol{\nu}) \\ &\leq \left(\frac{M \cdot N}{(\Gamma(M \cdot N + 1))^{\frac{1}{MN}}} \right)^{MN} \prod_{j=1}^{N} \prod_{i=1}^{M} \frac{1}{\rho \mu_i \nu_j} \\ &\quad \cdot \int_0^{\infty} \mathcal{Q}(\sqrt{2\gamma}) \frac{\gamma}{(\gamma)^{\frac{1}{MN}}} d\gamma \\ &= \frac{1}{2} \frac{(MN)^{MN-1}}{(\Gamma(M \cdot N + 1))} \left(\frac{1}{\rho} \right)^{MN} \prod_{j=1}^{N} \prod_{i=1}^{M} \frac{1}{\mu_i \nu_j}. \end{aligned} \tag{10}$$

Using the upper bound in (10), the diversity advantage is easily computed as $d = M \cdot N$. This completes the proof. ■

IV. Examples of Correlation Scenarios

A. Completely Uncorrelated Case

In the case in which the transmit and receive antennas are completely uncorrelated, i.e. $\mathbf{R}_T = \mathbf{R}_R = \mathbf{I}$, the pdf of the instantaneous signal-to-noise ratio (SNR) γ is given by

$$p(\gamma) = \left(\frac{\gamma}{\rho} \right)^{K-1} \cdot e^{-\frac{\gamma}{\rho}} \cdot \frac{1}{\Gamma(K) \rho} \tag{11}$$

with $K = n_T \cdot n_R$. This is a central χ_2 distribution with $2K$ degrees of freedom. The BER in (4) can be computed in closed form and it follows

$$\mathrm{BER}_{uc}(\rho, n_T, n_R) = \frac{1}{2} - \frac{2\rho \mathcal{H}\left(\left[\frac{1}{2}, \frac{1}{2} + n_T n_R\right], \frac{3}{2}, -\rho\right) \Gamma\left(\frac{1}{2} + n_T n_R\right)}{\sqrt{\pi} \Gamma(n_T n_R)} \tag{12}$$

with the hypergeometric function $\mathcal{H}$. This result is a special case of the result in [5] for BPSK. The diversity advantage is given by $d_{uc} = n_T \cdot n_R$.

B. Completely Correlated Case

In the case in which the transmit and receive antennas are completely correlated, i.e. $\mu_1 = n_T, \mu_2 = \ldots = \mu_{n_T} = 0$ and $\nu_1 = n_R, \nu_2 = \ldots = \nu_{n_R} = 0$ the pdf of the instantaneous SNR γ is given by

$$p(\gamma) = \frac{e^{\frac{-\gamma}{\rho n_T n_R}}}{\rho n_T n_R}. \tag{13}$$

This is a exponential distribution. The BER in (4) can be easily computed in closed form and it follows that

$$\text{BER}_{cc}(\rho, n_T, n_R) = \frac{1}{2} - \frac{1}{2\sqrt{1 + \frac{1}{\rho n_T n_R}}}. \tag{14}$$

From (14), the diversity advantage d is easily computed as $d_{cc} = 1$.

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Here, consider the case with $n_T = 2$ and $n_R = 1$. In the following, we write $\mu_2 = 2 - \mu_1$ and assume that $\mu_1 \neq \{0, 1, 2\}$. Then the pdf of the instantaneous SNR γ is given by

$$p(\gamma) = \frac{1}{2} \frac{e^{-\frac{\gamma}{\rho(2-\mu_1)}} - e^{-\frac{\gamma}{\rho \mu_1}}}{(\mu_1 - 1)\rho}. \tag{15}$$

The BER in (4) can be computed in closed form and it follows

$$\begin{aligned} \text{BER}_{al}(\rho) = \frac{1}{2} - \frac{1}{4} \frac{\mu_1}{(\mu_1 - 1)\sqrt{1 + \frac{1}{\rho \mu_1}}} \\ + \frac{1}{4} \frac{(2 - \mu_1)}{(\mu_1 - 1)\sqrt{1 + \frac{1}{\rho(2-\mu_1)}}}. \end{aligned} \tag{16}$$

From (16), the diversity advantage d is computed as $d_{al} = 2$.

V. Numerical Simulation

In Fig. 1, the analytical and simulated BER results are compared for OSTBC with $n_T = 2, 4$ transmit and $n_R = 1, 2, 3$ receive antennas. The analytical results fit the simulated curves correctly. We observe that OSTBC are very robust against spatial correlation at the transmitter and the receiver. If the correlation matrices $\mathbf{R}_R$ and $\mathbf{R}_T$ have full rank, the BER curve has the same diversity but is shifted to the right. Only if the correlation matrices have not full rank, the diversity is reduced.

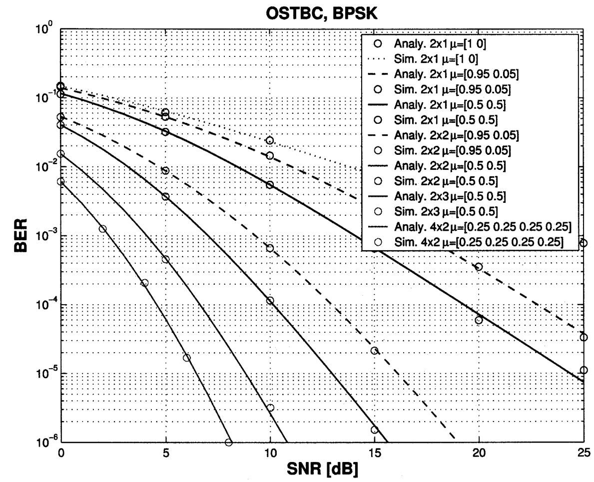

Fig. 1. Comparison of analytical and simulated BER of OSTBC's with different levels of correlation and different numbers of transmit and receive antennas.

VI. Conclusion

We analyzed the impact of spatial correlation on the performance of OSTBC. The exact BER formula under correlation at the transmitter and the receiver was derived. For some special cases, this lead to simple expressions of the BER. We have shown that OSTBC are robust against spatial correlation and experience a diversity advantage degradation only with singular correlation matrices.

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