Rate Region Frontiers for $n-$user Interference Channel with Interference as Noise

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Abstract— This paper presents the achievable rate region frontiers for the $n-$user interference channel when there is no cooperation at the transmit nor at the receive side. The receiver is assumed to treat the interference as additive thermal noise and does not employ multiuser detection. In this case, the rate region frontier for the $n-$user interference channel is found to be the union of n hyper-surface frontiers of dimension $n-1$, where each is characterized by having one of the transmitters transmitting at full power. The paper also finds the conditions determining the convexity or concavity of the frontiers for the case of two-user interference channel, and discusses when a time sharing approach should be employed with specific results pertaining to the two-user symmetric channel.

I. Introduction

The capacity region of a two-user communication channel has been an open problem for about 30 years [1], [2]. Information-theoretic bounds through achievable rate regions have been proposed, most famously with the Han-Kobayashi region [3]. The capacity of the Gaussian interference channel under strong interference has been found in [4]. Recent results on the two-user interference channel to within one bit of capacity have been shown in [5]. The aforementioned referenced literature focused on the two-user interference channel from an information-theoretic point of view. This work presents the frontiers for the achievable rate regions for $n-$user interference channel when the interference is treated as additive noise and no multiuser detection is employed. Examples where we encounter the need to define such rate regions are found in multicell communications, in addition to mesh and sensor networks where the preference is in employing low-complexity transceivers.

The system setup is presented in section II. Section III discusses the achievable rate frontiers for the two-user interference channel. The $n-$user generalization is treated in section IV. Section V focuses on characterizing the two-user interference channel in terms of convexity or concavity and when a time sharing approach should be used, with specific results to the symmetric channel.

II. System Setup

The $n-$user interference channel is presented in Fig. 1 with n transmitters and n receivers. The ith transmitter transmits its signal x_i to the intended i^{th} receiver with a power P_i. The receivers have independent additive complex white Gaussian
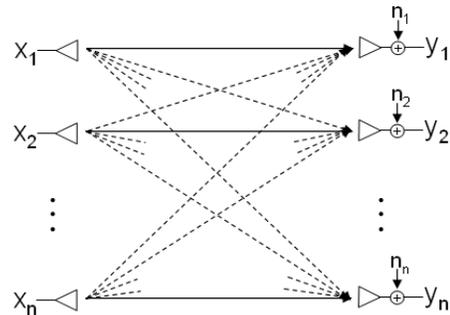
Fig. 1. $n-$user interference channel

noise with a power noise variance of σ_n^2. Each transmitter is assumed to have a maximum power constraint of $P_{\max}$. No cooperation is assumed between the nodes at the transmit side nor at the receive side. The transmitters have a single antenna each, and they communicate over frequency flat channels. $g_{i,j}$ denotes the channel power gain received at the ith receiver from the jth transmitter, and there are no constraints over the values or distributions of $g_{i,j}$. Therefore $g_{i,i}$ is the channel gain of the ith desired signal, where as $g_{i,j}$, $j \neq i$ represent the interfering channel gains. **P** is the transmit power vector of length n, where the ith element P_i denotes the transmit power of the ith transmitter. Treating the interference as additive noise throughout this paper and with no multiuser detection employed, C_i denotes the maximum reliable rate of communication between the ith transmitter and the ith receiver. Therefore the achievable rate for the ith transmit-receive pair is written as:

$$C_i(\mathbf{P}) = \log_2 \left(1 + \frac{g_{i,i} P_i}{\sigma_n^2 + \sum_{j \neq i} g_{i,j} P_j} \right). \qquad (1)$$

The objective of this work is to find the achievable rate region for the n transmit-receive pairs. Section III finds the achievable rate region frontier for the two-user channel, and section IV generalizes the frontier for the $n-$user case.

III. Achievable Rate Region Frontier for Two-user Interference Channel

This section studies the two-user interference channel. In this case, (1) can be expressed in function of P_1 and P_2 as $C_i(P_1, P_2)$, $i = 1, 2$. For notational brevity, the channel power
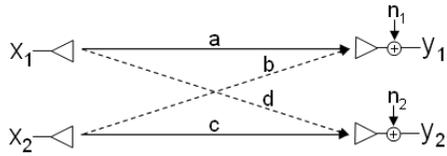
Fig. 2.   Two-user interference channel

gains are normalized by the noise variance, specifically:

$$a = g_{1,1}/\sigma_n^2, \qquad c = g_{2,2}/\sigma_n^2,$$
$$b = g_{1,2}/\sigma_n^2, \qquad d = g_{2,1}/\sigma_n^2.$$

The two-user interference channel is depicted in Fig. 2. C_1 and C_2 can therefore be written as:

$$C_1(P_1, P_2) = \log_2\left(1 + \frac{aP_1}{1 + bP_2}\right), \qquad (2)$$

$$C_2(P_1, P_2) = \log_2\left(1 + \frac{cP_2}{1 + dP_1}\right). \qquad (3)$$

Our objective is to find a frontier of the achievable rate region of (2) and (3) through the power control of P_1 and P_2, where each transmitter is subject to a maximum power constraint of $P_{\max}$.

Fig. 3 illustrates an example of rate region for a two-user interference channel, with C_1 and C_2 as the x-axis and the y-axis, respectively. From (2), C_1 in monotonically increasing in P_1 and monotonically decreasing in P_2, thus the point $C_1(P_{\max}, 0)$, alternatively annotated as point C on the x-axis in the figure, represents the maximum value C_1 can obtain. Similarly for the y-axis, the maximum value that C_2 achieves is $C_2(0, P_{\max})$, annotated as the point A on the y-axis. The point B has the following coordinates of $C_1(P_{\max}, P_{\max})$ and $C_2(P_{\max}, P_{\max})$.

A. Rate Region Frontier Formulation

The rate region frontier can be found by setting C_1 to a value R, then R is swept over the full range of C_1, i.e. from 0 to $C_1(P_{\max}, 0)$, while finding the maximum C_2 value that can be achieved for each R. Hence, for a constant rate $C_1 = R$,

$$C_1(P_1, P_2) = R = \log_2\left(1 + \frac{aP_1}{1 + bP_2}\right). \qquad (4)$$

Therefore the relation between P_1 and P_2 is obtained as follows:

$$P_1 = \frac{1}{a}(1 + bP_2)(2^R - 1). \qquad (5)$$

From (5), $C_2(P_1, P_2)$ can now be written in function of one parameter as $C_2(P_2)$, specifically:

$$C_2(P_2) = \log_2\left(1 + \frac{cP_2}{1 + \frac{d}{a}(1 + bP_2)(2^R - 1)}\right). \qquad (6)$$

It is important to analyze the behavior of $C_2(P_2)$ in terms of P_2. This is presented in the following lemma:

Lemma III.1 *Setting C_1 at a constant rate, $C_1(P_1, P_2) = R$, $C_2(P_2)$ is a monotonically increasing function in P_2.*
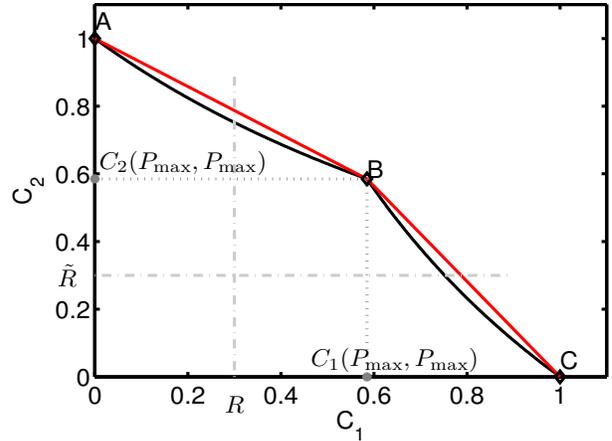

Fig. 3.   Two-user interference channel achievable rate region. (For $P_{\max} = 1$, $a = 1$, $b = 1$, $c = 1$, and $d = 1$)

Proof: The proof is provided in the Appendix. ∎

Remark: A direct implication of the monotonicity of the relation in (6) is that if C_2 is equal to a constant C_2^* for the rate of $C_1 = R$, then: i) there is a unique P_2^* that achieves C_2^*, ii) when P_2^* is determined, then $P_1 = P_1^*$ is uniquely defined from (5), iii) from P_1^* and P_2^*, C_1 is uniquely defined as $C_1 = R = C_1^*$ from (4). Thus P_1^* and P_2^* uniquely define a point in the rate region with coordinates C_1^* and C_2^*.

In other words, any point in the rate region is achieved solely by a unique power tuple. This leads to what we denote by potential lines Φ in the rate region which are formed by holding one of the power parameters constant to a certain value, and sweeping the other parameter over its full power range. For instance, the potential line $\Phi(:, P_{\max})$ is formed by sweeping P_1 from 0 to $P_{\max}$ and holding P_2 at $P_{\max}$. Based on the uniqueness property just discussed, these potentials lines along the P_1 dimension are therefore non-touching, i.e. $\Phi(:, p_2)$ and $\Phi(:, p_2')$ do not intersect if $p_2 \neq p_2'$.

The rate region frontier then simplifies to finding the maximum value $C_2(P_2)$ could achieve for any value of $C_1(P_1, P_2) = R$. Effectively, this is formulated as:

$$\begin{aligned} \arg\max_{P_2} \quad & C_2(P_2) \\ \text{subject to} \quad & C_1(P_1, P_2) = R \\ & P_i \leq P_{\max} \qquad i = 1, 2. \end{aligned} \qquad (7)$$

B. C_2 frontier for $0 \leq R \leq C_1(P_{\max}, P_{\max})$

As (4) is monotonically increasing in P_1 and monotonically decreasing in P_2, R can only exceed the value $C_1(P_{\max}, P_{\max})$ when P_2 is less than $P_{\max}$. Thus $P_2 = P_{\max}$ is attainable only when $0 \leq R \leq C_1(P_{\max}, P_{\max})$, and P_2 needs to be less than $P_{\max}$ otherwise. From the proof provided in Lemma III.1, where (6) is proved to be monotonically increasing in P_2, and for the following range of R: $0 \leq R \leq C_1(P_{\max}, P_{\max})$, the solution to (7) is:

$$\arg\max_{P_2} C_2(P_2) = P_{\max} \qquad (8)$$

Therefore in this range of R or equivalently of C_1, using (5) and (8), C_2 is expressed in function of C_1 as follows:

$$C_2(C_1) = \log_2\left(1 + \frac{cP_{\max}}{1 + \frac{d}{a}(1 + bP_{\max})(2^{C_1} - 1)}\right).$$

C. C_2 frontier for $C_1(P_{\max}, P_{\max}) \leq R \leq C_1(P_{\max}, 0)$

Using symmetry of the previous result, for a constant rate $C_2 = R$, there is a linear relation between P_1 and P_2. And thus $C_1(P_1, P_2)$ can be written in function of one parameter P_1 as follows:

$$C_1(P_1) = \log_2\left(1 + \frac{aP_1}{1 + \frac{b}{c}(1 + dP_1)(2^{\tilde{R}} - 1)}\right).$$

And by symmetry of the result in Lemma III.1, $C_1(P_1)$ is monotonically increasing in P_1. Thus by symmetry, for the following range of $\tilde{R}$:

$$0 \leq \tilde{R} \leq C_2(P_{\max}, P_{\max}), \tag{9}$$

we have:

$$\arg\max_{P_1} C_1(P_1) = P_{\max}.$$

Therefore for this range of $\tilde{R}$, $P_1 = P_{\max}$ is attainable and maximizes $C_1(P_1)$. Correspondingly, C_1 spans the following range:

$$C_1(P_{\max}, P_{\max}) \leq C_1 \leq C_1(P_{\max}, 0). \tag{10}$$

So for the range of $\tilde{R}$ in (9) and the range of C_1 in (10), $P_1 = P_{\max}$ describes the frontier. Therefore the values of C_1 at the frontier are:

$$C_1(P_{\max}, P_2) = \log_2\left(1 + \frac{aP_{\max}}{1 + bP_2}\right).$$

Hence for $C_1(P_{\max}, P_{\max}) \leq R \leq C_1(P_{\max}, 0)$, the value of P_2 that will achieve the frontier follows as:

$$P_2 = \frac{1}{b}\left(\frac{aP_{\max}}{2^R - 1} - 1\right). \tag{11}$$

So effectively the value found in (11) is the answer for (7) for the range of $C_1(P_{\max}, P_{\max}) \leq R \leq C_1(P_{\max}, 0)$.

D. Achievable Rate Region Frontier

This subsection consolidates the two results to fully describe the rate region frontier. For a value of c_1 that sweeps the full range of C_1, we have:

- for $0 \leq c_1 \leq C_1(P_{\max}, P_{\max})$

$$\arg\max_{P_2} C_2(P_2) = P_{\max}$$

and the frontier, denoted by $\mathcal{F}_2 = \Phi(:, P_{\max})$, is expressed as:

$$C_2(c_1) = \log_2\left(1 + \frac{cP_{\max}}{1 + \frac{d}{a}(1 + bP_{\max})(2^{c_1} - 1)}\right) \tag{12}$$
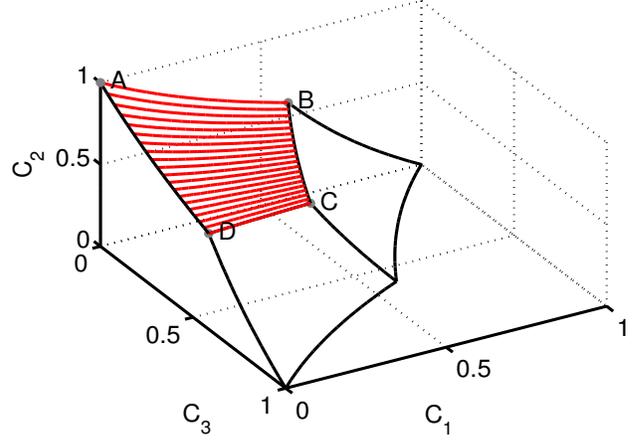

Fig. 4.   A 3-user interference channel rate region

- for $C_1(P_{\max}, P_{\max}) \leq c_1 \leq C_1(P_{\max}, 0)$

$$\arg\max_{P_2} C_2(P_2) = \frac{1}{b}\left(\frac{aP_{\max}}{2^{c_1} - 1} - 1\right)$$

and the frontier, denoted by $\mathcal{F}_1 = \Phi(P_{\max}, :)$, is expressed as:

$$C_2(c_1) = \log_2\left(1 + \frac{\frac{c}{b}(aP_{\max} - (2^{c_1} - 1))}{(2^{c_1} - 1)(1 + dP_{\max})}\right). \tag{13}$$

The notation $\mathcal{F}_i$ denotes a potential line parameterized by holding the ith element in the power tuple at maximum power.

Finally, the rate region frontier $\mathcal{F}$ for a two-user interference channel is obtained as:

$$\mathcal{F} = \text{Convex Hull}\{\mathcal{F}_1 \cup \mathcal{F}_2\}. \tag{14}$$

The convex hull operation stems from the time sharing solution of the extremity points in the frontiers in order to arrive to a convex rate region. For example, in Fig. 3, $\mathcal{F}$ is described by connecting points A and B, and points B and C.

IV. ACHIEVABLE RATE REGION FRONTIER FOR $n-$USER INTERFERENCE CHANNEL

This section starts by considering a $3-$user interference channel to show the effect of adding a new dimension, then generalizes the results for the $n-$user case.

A. $3-$user example: Effect of increasing P_3 from 0 to $P_{\max}$

The rate region for the $3-$user case is illustrated in Fig. 4. The following notation of $\Phi(P_1, P_2, P_3)$ denotes a point in the rate region with coordinates of $[C_1(P_1, P_2, P_3), C_2(P_1, P_2, P_3), C_3(P_1, P_2, P_3)]$. Accordingly, $\Phi(:, P_{\max}, P_3)$ describes a line characterized by sweeping the transmit power of the first transmitter P_1 from 0 to $P_{\max}$, with the second transmitter transmitting at $P_{\max}$, and the third transmitter transmitting at a value of P_3. Similarly, $C_i(:, P_{\max}, :)$ represents a surface in the rate region marked

by sweeping the full range of P_1 and P_3, and holding P_2 at $P_{\max}$.

When $P_3 = 0$, the same setup and results that were described in section III applies. Specifically, for the rate range of $0 \leq C_1 \leq C_1(P_{\max}, P_{\max}, 0)$ and $0 \leq C_2 \leq C_2(0, P_{\max}, 0)$ and $C_3 = 0$, the frontier can be described as $\Phi(:, P_{\max}, 0)$, which is the line from point A to point B in Fig. 4. As P_3 increases, we want to describe the subsequent effect and how it is traced in the rate region.

Revisiting the equation in (1), a constant P_3 has the effect of just an additive noise term in $C_1(\mathbf{P})$ and $C_2(\mathbf{P})$. Hence, all the previous results in section III are applicable for any value of P_3 in describing the frontier for C_1 and C_2; as the effect of P_3 can be lumped in the noise term. Thus for the range of $0 \leq C_1 \leq C_1(P_{\max}, P_{\max}, P_3)$ and $0 \leq C_2 \leq C_2(0, P_{\max}, P_3)$, where P_3 is constant, the frontier line on C_1 and C_2 is $\Phi(:, P_{\max}, P_3)$, i.e. characterized by having $P_2 = P_{\max}$. Consequently, the potential lines (or surfaces) concept in the 3$-$user case carries through.

Next, the frontier on C_3 needs to be described. For each value of P_3, $\Phi(:, P_{\max}, P_3)$ traces one of the highlighted curves in Fig. 4. For these collection of lines to form a frontier we want to prove that at each increasing value of P_3 these potential lines monotonically increase in the C_3 dimension. This is obvious from the C_3 and P_3 relation in (1). The maximum value of C_3 that can be achieved in this case is when $P_3 = P_{\max}$, i.e. $C_3(:, P_{\max}, P_{\max})$. Therefore the highlighted frontier surface in Fig. 4 is the potential surface $\Phi(:, P_{\max}, :)$. The boundary contours of this surface are the potential lines: $A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, and $D \leftrightarrow A$, defined as $\Phi(:, P_{\max}, 0)$, $\Phi(P_{\max}, P_{\max}, :)$, $\Phi(:, P_{\max}, P_{\max})$, and $\Phi(0, P_{\max}, :)$, respectively.

By symmetry of interchanging P_1, P_2 and P_3, we find that the 3$-$user rate region frontier is determined through the union of three surfaces: $\Phi(P_{\max}, :, :) \cup \Phi(:, P_{\max}, :) \cup \Phi(:, :, P_{\max})$. And $\mathcal{F}$ is expressed as:

$$\mathcal{F} = \text{Convex Hull}\{\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3\}, \tag{15}$$

where $\mathcal{F}_i$ is a potential surface $\Phi(\cdot)$ with $P_{\max}$ in the ith power position. (Note that the intersection of potential surfaces is a potential line, as two of the dimensional inputs become equal, i.e. $\Phi(P_{\max}, P_{\max}, :) \in \mathcal{F}_1$ and $\Phi(P_{\max}, P_{\max}, :) \in \mathcal{F}_2$.)

B. $n-$user generalization

The case for $n-$user generalization can be done by induction. For the nth added dimension to the existing $n-1$ dimensions problem, the additional power effect of P_n can be lumped in the additive noise term of the existing expressions, and thus the results for $C_1, \ldots, C_{n-1}$ hold and carry through. The frontier on C_n is monotonically increasing in P_n, and can be maximized with $P_n = P_{\max}$ for the appropriate range in $C_1, \ldots, C_{n-1}$. Invoking symmetry we can generalize over all the rate ranges, therefore arriving to the following theorem.
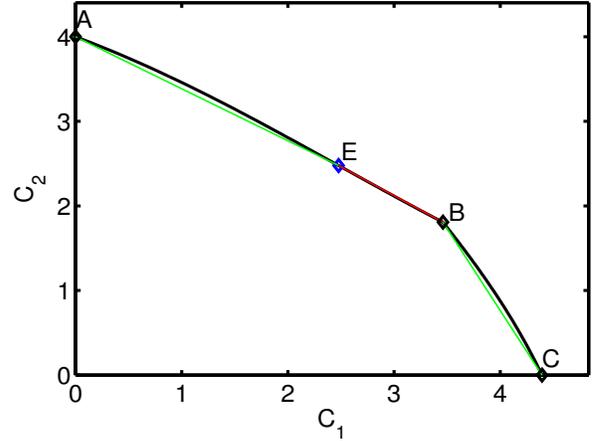

Fig. 5. Non-stationary inflection point E on $\mathcal{F}_2$ and concavity on $\mathcal{F}_1$. ($Q_1 = 0.45$ and $Q_2 = 3.11$, for $P_{\max} = 1$, $a = 20$, $b = 1$, $c = 15$, $d = 5$.)

Theorem IV.1 *The achievable rate region frontier of the $n-$user channel is:*

$$\mathcal{F} = \text{Convex Hull}\{\cup_{i=1}^{n} \mathcal{F}_i\}, \tag{16}$$

where $\mathcal{F}_i$ is a hyper-surface of $n-1$ dimensions characterized by holding the ith transmitter at full power $P_{\max}$.

Using the notation introduced in this section, $\mathcal{F}_i$ is effectively $\Phi(:, \ldots, P_{\max}, \ldots, :)$ with $P_{\max}$ at the ith power position.

V. CHARACTERISTICS OF THE ACHIEVABLE RATE REGION FOR TWO-USER INTERFERENCE CHANNEL

Treating the two-user interference channel in more details, this section studies the behavior of the rate region frontiers $\mathcal{F}_1$ and $\mathcal{F}_2$ in terms of convexity and concavity. In addition, it discusses when a time-sharing approach should be employed, with specific results pertaining to the symmetric channel.

A. Convexity or Concavity of the Frontiers

The frontier $\mathcal{F}_2$ in (12) depends on P_1 through the following relation of c_1 and P_1:

$$P_1 = \frac{1}{a}(1 + bP_{\max})(2^{c_1} - 1).$$

Therefore the second derivative of $\mathcal{F}_2$ with respect to c_1 leads to the following expression:

$$\frac{\partial^2 \mathcal{F}_2}{\partial c_1^2} = (\theta + adP_1)^2 - (a - \theta)(a - \theta + acP_{\max}),$$

where $\theta = d + dbP_{\max}$. Therefore if the frontier line is concave (i.e. $\frac{\partial^2 \mathcal{F}_2}{\partial c_1^2} \leq 0$) then the enclosed region is convex, i.e. the straight line connecting any two points in the rate region is entirely enclosed in the rate region. Let $\Re(\cdot)$ be the real operation, and defining the quantity Q_1 as:

$$Q_1 = \frac{\Re(\sqrt{(a - \theta)(a - \theta + acP_{\max})}) - \theta}{ad}, \tag{17}$$

then it suffices to study the convexity or concavity by examining the sign of $(P_1 - Q_1)$, where Q_1 is derived such as:

$$\text{sign}\left(\frac{\partial^2 \mathcal{F}_2}{\partial c_1^2}\right) = \text{sign}(P_1 - Q_1).$$

Thus the convexity or concavity of the frontier line $\mathcal{F}_2$ is governed by:

- $Q_1 \leq 0$: the frontier line $\mathcal{F}_2$ is convex, and the region bounded by $\mathcal{F}_2$ is concave. As $P_1 - Q_1 \geq 0$ for all the range of P_1.
- $Q_1 \geq P_{\max}$: the frontier line $\mathcal{F}_2$ is concave, and the region bounded by $\mathcal{F}_2$ is convex. As $P_1 - Q_1 \leq 0$ for all the range of P_1.
- $0 < Q_1 < P_{\max}$: the frontier line exhibits a non-stationary inflection point when $P_1 = Q_1$, and $\mathcal{F}_2$ is neither convex nor concave between the point extremities of $\Phi(0, P_{\max})$ and $\Phi(P_{\max}, P_{\max})$. In this case:
    - for $0 < P_1 \leq Q_1$: the line $\Phi(0 : Q_1, P_{\max})$ is concave, i.e. the frontier segment between point $\Phi(0, P_{\max})$ (the point A in Fig. 5) and point $\Phi(Q_1, P_{\max})$ (the point E in Fig. 5) is concave.
    - for $Q_1 \leq P_1 < P_{\max}$: the line $\Phi(Q_1 : P_{\max}, P_{\max})$ is convex, i.e. the frontier segment between point $\Phi(Q_1, P_{\max})$ (the point E in Fig. 5) and point $\Phi(P_{\max}, P_{\max})$ (the point B in Fig. 5) is convex.

By symmetry, the frontier line $\mathcal{F}_1$ exhibits the following behavior: it is convex when $Q_2 \leq 0$, and it is concave when $Q_2 \geq P_{\max}$, and it exhibits an non-stationary inflection point when $P_2 = Q_2$ – specifically it is convex for $0 < P_2 \leq Q_2$ and concave for $Q_2 \leq P_2 < P_{\max}$. Hereby Q_2 is defined as:

$$Q_2 = \frac{\Re(\sqrt{(c - \beta)(c - \beta + acP_{\max})}) - \beta}{cb}, \quad (18)$$

with $\beta = (b + bdP_{\max})$.

When describing the full rate region frontier through $\mathcal{F}_1 \cup \mathcal{F}_2$, the rate region is convex if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are concave, and the rate region is concave otherwise. Therefore, whenever the frontier (or segment thereof) is concave, it will describe the convex hull of the rate region instead of a time sharing solution. Fig. 5 illustrates an example where the frontier $\mathcal{F}_1$ is concave, and the frontier $\mathcal{F}_2$ exhibiting a non-stationary inflection point E. In this case the convex hull rate region is found by operating along the concave frontier $\mathcal{F}_1$, and time-sharing between point B and point E, and operating along the concave segment of $\mathcal{F}_2$ between point E and point A.

B. Optimality of Time Sharing

This subsection investigates the optimality of time sharing between operating points in the rate region. For instance, whenever the rate region frontier segment is convex (equivalently, the enclosed rate region is concave) then operating with time sharing between the extremities of the curve is optimal than operating along the log-defined frontier. Analyzing the $\mathcal{F}_2$ frontier, and referring to the Fig. 3, it follows:

- $Q_1 \leq 0$: (i.e. $\mathcal{F}_2$ frontier is convex) it is optimal to apply time-sharing through the following options:
    - between point A and point B,
    - between point A and point $\Phi(P_{\max}, Q_2)$ if $\mathcal{F}_1$ exhibits a non-stationary inflection point,
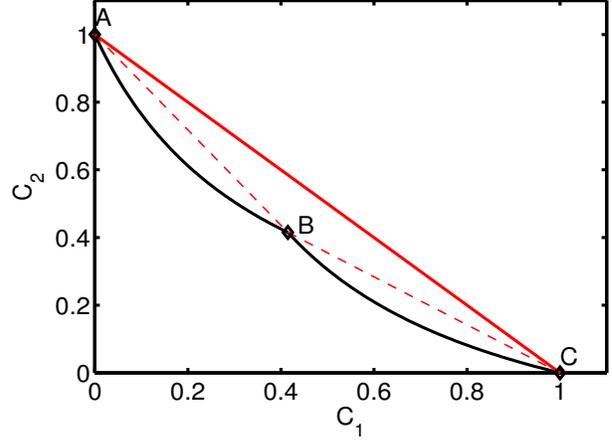   - between point A and a point on the concave segment of $\mathcal{F}_1$,

Fig. 6. Optimal time sharing between point A and point C. (For $P_{\max} = 1$, $a = 1$, $b = 2$, and b* calculated from (20) equals $\sqrt{2}$.)

    - between point A and point C.
    These depend on how the parameters a, b, c, d, and $P_{\max}$ would lead to a convex hull region. This can be done by evaluating and comparing each of the candidate solution aforementioned.
- $Q_1 \geq P_{\max}$: (i.e. $\mathcal{F}_2$ frontier is concave) the potential line $\Phi(:, P_{\max})$ is optimal, and no time sharing is to be employed.
- $0 < Q_1 < P_{\max}$: it is optimal to use the concave potential line segment $\Phi(0 : Q_1, P_{\max})$, and subsequently to use the time sharing candidate options that the case $Q_1 \leq 0$ mentioned but with replacing the point A with the point $\Phi(Q_1, P_{\max})$.

The conditions for $\mathcal{F}_1$ follow by symmetry.

1) Time sharing between Points A, B, and C in Fig. 3: Discounting the case when $\mathcal{F}_1$ or $\mathcal{F}_2$ exhibit non-stationary inflection point for simplicity, and focusing on the case of $Q_1 \leq 0$ and $Q_2 \leq 0$, it is important to know when time sharing between point A and point C is better than time sharing through the intermediate point B. This is done by comparing the straight line connecting points A and C, and the coordinates of B. It follows that operating with time sharing between the points (or system states) A and C (i.e. one transmitter only transmitting at a certain point) is optimal when:

$$\frac{(1 + cP_{\max})(1 + dP_{\max})}{1 + cP_{\max} + dP_{\max}} \geq \left(\frac{1 + aP_{\max} + bP_{\max}}{1 + bP_{\max}} \right)^{\gamma} \quad (19)$$

with $\gamma = \log_2(1 + cP_{\max}) / \log_2(1 + aP_{\max})$.

2) Symmetric two-user interference channel: For the symmetric two-user interference channel, $a = c$, and $b = d$. In this case, (19) simplifies and leads to the following theorem:

Theorem V.1 *Time-sharing operation with one transmitter active at full power at a time is optimal when*

$$b \geq \frac{\sqrt{1 + aP_{\max}}}{P_{\max}}. \quad (20)$$

Remark: For high SNR (i.e. $aP_{\max} \gg 1$), (20) reduces to $bP_{\max} \geq \sqrt{aP_{\max}}$, which interestingly coincides with recent results in [5, eq.(3)].

This indicates that when the cross interfering power gain b exceeds the right hand side value in (20), denoted by b^*, it is optimal to operate with one transmitter at a time. Fig. 6 uses $b = 2$, which is larger than $b^* = \sqrt{2}$ obtained from (20) for $a = 1$ and $P_{\max} = 1$. By contrast in Fig. 3, the value of $b = 1$ is adopted and the behavior exhibited is different as expected.

In addition, we subsequently prove that the expression in (20) is a sufficient condition for both frontiers $\mathcal{F}_1$ and $\mathcal{F}_2$ to be convex, i.e. Q_1 and Q_2 are always ≤ 0.

Proof: For the symmetric case, $Q_1 = Q_2 = Q_{sym}$ can be written as

$$Q_{sym} = \frac{\Re(\sqrt{(a-\theta)(a-\theta+a^2 P_{\max})}) - \theta}{ab},$$

where $\theta = b + b^2 P_{\max}$. Q_{sym} can also be written in this form:

$$Q_{sym} = \frac{\Re(\sqrt{T_1 T_2}) - \theta}{ab},$$

where $T_1 = a - \theta = a - b - b^2 P_{\max}$, and $T_2 = a - \theta + a^2 P_{\max}$. From the expression in (20), a can be alternatively upper-bounded as $a \leq (b^2 P_{\max} - 1/P_{\max})$. Therefore, T_1 is upper-bounded as:

$$T_1 \leq -1/P_{\max} - b. \tag{21}$$

From (21), T_1 is always negative. T_2 however can be positive or negative. Each case is evaluated as follows:

- $T_2 \geq 0$: then $\Re(\sqrt{T_1 T_2}) = 0$, and as θ is always positive, then $Q_{sym} \leq 0$.
- $T_2 \leq 0$: $\Re(\sqrt{T_1 T_2})$ is ≥ 0. In this case, the numerator of Q_{sym} can be written as:

$$\text{num}(Q_{sym}) = \sqrt{(\theta-a)(\theta-a-a^2 P_{\max})} - \theta.$$

Given the fact that $(\theta - a - a^2 P_{\max}) \leq (\theta - a)$, then $\text{num}(Q_{sym})$ can be upper-bounded as:

$$\begin{aligned} \text{num}(Q_{sym}) &\leq \sqrt{(\theta-a)^2} - \theta \\ &\leq -a \leq 0. \end{aligned}$$

Hence the frontiers $\mathcal{F}_1$ and $\mathcal{F}_2$ are convex. ∎
Therefore, when b satisfies the equation in (20) the frontiers lines are always convex, and time-sharing with only one transmitter active at a time is optimal.

VI. CONCLUSIONS

The achievable rate region frontiers for the $n-$user interference channel were presented when there is no cooperation at the transmit side nor at the receive side. The receivers do not employ multiuser detection, and the interference is considered as additive noise. Results were first found for the two-user interference channel. The $3-$user interference channel was treated next to show the effect of adding the additional dimension, and subsequently the $n-$user interference channel generalization results followed. The $n-$user rate region is found to be the convex hull of the union of n hyper-surfaces

each of dimension $n - 1$. Each hyper-surface frontier $\mathcal{F}_i$ is defined by having the ith transmitter transmitting at its full power $P_{\max}$.

The two-user interference channel was further studied regarding the convexity or the concavity of the frontiers. Conditions when the frontier is convex or concave or exhibiting a non-stationary inflection point were also obtained. Whenever the log-defined frontier line is convex then a time sharing solution is optimal. For the symmetric two-user case, the condition was found to indicate when time-sharing between the points $\Phi(P_{\max}, 0)$ and $\Phi(0, P_{\max})$ (i.e. one transmitter solely transmitting at full power at a certain time) is to be used, rather than time-sharing through the point $\Phi(P_{\max}, P_{\max})$ (i.e. both transmitters transmitting at full power). That condition was also proven to be sufficient to ensure that both frontiers ($\mathcal{F}_1$ and $\mathcal{F}_2$) will in fact always be convex.

APPENDIX

Proof that the equation (6), $C_2(P_2)$ is monotonically increasing in P_2.
Proof: Effectively (6) is in the form of $f(1 + g(x))$. As $f(\cdot)$ is monotonically increasing in its argument, it suffices to prove that $g(x)$ is monotonically increasing in x. Therefore define $g(P_2)$ as:

$$g(P_2) = \frac{acP_2}{a + d(1 + bP_2)(2^R - 1)},$$

$$\frac{\partial g(P_2)}{\partial P_2} =$$

$$\begin{aligned} &\frac{ac}{a + d(1 + bP_2)(2^R - 1)} - \frac{acP_2 db(2^R - 1)}{(a + d(1 + bP_2)(2^R - 1))^2} \\ &= \frac{a^2 c + acd(1 + bP_2)(2^R - 1) - acP_2 db(2^R - 1)}{(a + d(1 + bP_2)(2^R - 1))^2} \\ &= \frac{a^2 c + acd(2^R - 1) + acdbP_2(2^R - 1) - acdbP_2(2^R - 1)}{(a + d(1 + bP_2)(2^R - 1))^2} \\ &= \frac{ac(a + d(2^R - 1))}{(a + d(1 + bP_2)(2^R - 1))^2}. \end{aligned} \tag{22}$$

The numerator in (22) is $\neq 0$ if $a \neq 0$ and $c \neq 0$ ($a = 0$ or $c = 0$ are the trivial cases where the rate region is either a line or the point zero). As $R \geq 0$, then $(2^R - 1) \geq 0$. Thus $\partial g(P_2)/\partial P_2$ is always > 0 for non-trivial cases of a and c. Thus $g(P_2)$ is monotonically increasing in P_2, and equivalently $C_2(P_2)$ is monotonically increasing in P_2. ∎

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