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# An Extension of Darlington's Algorithm for the Design of Elliptic Filters

Karlheinz Ochs and Michael Vollmer

***Abstract*—For the computation of parameters of elliptic filters, the evaluation of elliptic functions is necessary. For this, one can use numerical methods like Landen's transformation or Darlington's algorithm. In this paper, Darlington's algorithm is extended and connected with Landen's transformation. All formulae are mainly derived for symmetric and antimetric wave digital filters having either an even or an odd characteristic function.**

***Index Terms*—Antimetric, Darlington's algorithm, elliptic digital filters, Landen's transformation, lossless two-ports, symmetric, wave digital filters.**

## I. INTRODUCTION

In the field of digital signal processing, one is often faced with the design of digital filters with extreme high requirements. These requirements concern a low attenuation in the passband as well as a high attenuation in the stopband in addition to a narrow transition band. It is well known that if there are no restrictions for the phase or the group delay, respectively, these specifications are met with minimum order by so-called elliptic filters [1]. This designation results from the necessary evaluation of elliptic functions in order to obtain the filter parameters. Since an analytic evaluation of elliptic functions is not possible we have to deal with numerical methods [2], [11], [12]. For the aim of this paper, we restrict ourselves to the so-called Landen transformation. Beside this well-known numerical method, Darlington has proposed a special algorithm [3] for the design of elliptic filters which was extended by Bàez-Lòpez [4].

This paper is devoted to a derivation of Darlington's formulae as well as to an extension of the existing concept by using the ascending Landen transformation. For this purpose, it has been an important result to achieve a high similarity of the new transformation formulae with Darlington's formulae.

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## II. CHARACTERISTIC FUNCTION OF AN ELLIPTIC FILTER

It is well-known that lossless 2-ports possess two power complementary transfer functions—a circumstance that makes them extremely suitable for usage as branching networks if they are realized as digital filters. The scattering matrix $\boldsymbol{S}(\psi)$[1] of such a 2-port is essentially equivalent to a matrix having the form

$$\boldsymbol{S}(\psi) = \begin{bmatrix} S_{11}(\psi) & S_{21}(\psi) \\ S_{21}(\psi) & S_{11}(\psi) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} S_1(\psi) & 0 \\ 0 & S_2(\psi) \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{1}$$

which can be realized with the canonical reflectances $S_1$ and $S_2$, cf. [6], [7], and [10]. In this correspondence, we will consider symmetric and antimetric 2-ports only.

Now, a so-called characteristic function is introduced, i.e., $C \stackrel{\text{def}}{=} S_{11}/S_{21}$. The denominator polynomial of $C$ is the same as the numerator polynomial of the transmittance. Since the 2-port is lossless, $S_{21}$ can be reformulated as

$$|S_{21}(j\varphi)|^2 = 1/[1 + |C(j\varphi)|^2]. \tag{2}$$

If the branching network is assumed to be a lowpass and a highpass filter, we can alternatively formulate the demands as

$$|C(j\varphi)| \le C_{\max} \quad \text{for } |\varphi| \le \varphi_c$$

and

$$|C(j\varphi)| \ge C_{\min} \quad \text{for } |\varphi| \ge \varphi_s$$

for the magnitude of the characteristic function. From the approximation theory, it is well-known that for an optimal solution, all zeros $\varphi_{0\nu}$ and poles $\varphi_{\infty\mu}$ have to lie on the imaginary axis in the interval $|\varphi| \le \varphi_c$ and $|\varphi| \ge \varphi_s$, respectively. In this case, one easily checks that the numerator and denominator polynomial have no common zeros.[2] Owing to this, the characteristic function can be written as

$$C(\psi) = M\psi^{\delta_n}\prod_{\mu=1}^{m}(1+\psi^2/\varphi_{0\mu}^2)/(1+\psi^2/\varphi_{\infty\mu}^2)$$
$$|\varphi_{0\nu}| < \varphi_c, \quad |\varphi_{\infty\mu}| > \varphi_s$$

with the real parameters $\varphi_{0\mu}$ and $\varphi_{\infty\mu}$ and the real or imaginary constant $M$. Here,$\delta_n + \overline{\delta}_n \stackrel{\text{def}}{=} 1$ with $\delta_n \stackrel{\text{def}}{=} 0$ for $n$ even and $\delta_n \stackrel{\text{def}}{=} 1$ for $n$ odd has been defined in order to support a convenient notation for the different cases.

[1]For digital filters that operate at an operating period $T$, it is convenient to use the equivalent complex frequency $\psi \stackrel{\text{def}}{=} \tanh(pT/2)$ instead of using the usual complex frequency $p$. The imaginary part of $\psi$ is the so-called equivalent real frequency $\varphi \stackrel{\text{def}}{=} \text{Im}\ \psi = \tan(\omega T/2)$, which maps the Nyquist interval $-T/\pi < \omega < T/\pi$ bijectively onto the interval $-\infty < \varphi < \infty$.

[2]Since the numerator polynomial of $S_{21}$ has only zeros on the imaginary axis, whereas the denominator polynomial is a Hurwitz polynomial having all zeros lying in the left half plane, the transfer functions cannot inhere allpass factors. In consequence, $S_{21}$ is of minimum phase.

Reformulating this problem according to Cauer [1], the elliptic differential equations

$$\begin{aligned}&\mathrm{d}C\sqrt{[1-(\varphi/\varphi_c)^2]\,[1-k^2(\varphi/\varphi_c)^2]}\\&\quad=\tilde{M}\,d\varphi\sqrt{[1+(C/C_{\max})^2]\,[1+g^2(C/C_{\max})^2]}\end{aligned}\tag{3}$$

are obtained with the moduli $g\overset{\text{def}}{=}C_{\max}/C_{\min}$ and $k\overset{\text{def}}{=}\varphi_c/\varphi_s$.

From now on, we will agree with the convention that a complementary modulus is designated in the same way as the modulus with an additional prime: $k^2+k'^2=1$. Furthermore, each complete elliptic integral is denoted by a capital letter corresponding to the related modulus.

The solution of the differential equation can be found if (3) is integrated where the edge conditions have to be taken into account [1], [9]:

$$C(\psi)=jC_{\max}\mathrm{sn}(nuG+\overline{\delta}_nG,g)$$

and

$$\psi=j\,\varphi_c\mathrm{sn}(uK,k).\tag{4}$$

Additionally, the relation between the complete elliptic integrals $KG'=nK'G$ has to be satisfied because $C(\psi)$ must be a rational function.

Finally, taking some fundamental relations of elliptic functions into account, we may express the characteristic function as

$$C(\psi)=jC_{\min}\,\mathrm{dc}(nuG,g)\quad\text{and}\quad\psi=j\,\varphi_s\,\mathrm{dc}(uK,k).\tag{5}$$

## III. Extension of Darlington's Algorithm

As will be shown, Darlington's algorithm can be obtained by applying the descending Landen's transformation. The advantage of his algorithm is the ability to interpret the formulae of each iteration step as a further characteristic function of an elliptic filter. After four transformation steps, his numerical method in order to compute the poles of the characteristic function is analogous to the computation of the poles of a Chebyshev filter.

If the descending Landen transformation with a fixed number of iteration steps like $R=S=4$ is used, Darlington's algorithm [3] is obtained.

Since Darlington's algorithm is based on descending Landen transformation, it yields an unnecessary effort for the design of filters with a narrow transition band. In order to tackle this problem, Darlington's algorithm is generalized to a certain extent.

### A. Landen's Transformation

The principle of Landen's transformation is to replace an elliptic function with another elliptic function (which has a different modulus) where both functions are related via a simple algebraic equation. With a chain of such transformation steps, we are able to track the modulus toward zero. Then, the transformed elliptic function can be approximated by a trigonometric function. Tracking back this quantity, the approximation value of the elliptic function is obtained. This kind of transformation is called descending Landen transformation. The ascending Landen transformation follows a similar chain of transformation steps.

According to the descending Landen transformation [8]

$$\begin{aligned}\mathrm{sn}(u_{r-1},k_{r-1})&=\frac{(1+k_r)\mathrm{sn}(u_r,k_r)}{1+k_r\mathrm{sn}^2(u_r,k_r)}\\u_r&=\frac{u_{r-1}}{1+k_r},\quad k_r=\frac{1-k'_{r-1}}{1+k'_{r-1}}\end{aligned}\tag{6}$$

a transformation for $\mathrm{dc}(uK_r,k_r)$ will be derived. For this, a relation between the complete elliptic integrals is needed, which can be found by inserting $u_r=K_r$ in (6), where $\mathrm{sn}(K,k)=1$ has to be considered:

$$(1+k_r)K_r=K_{r-1}.$$

Using the identity $\mathrm{dc}(u-K,k)=1/\mathrm{sn}(u,k)$ (cf. [8]) and describing the complementary modulus $k'$ in dependence of the associated modulus $k$, we get (6) in another guise:

$$\begin{aligned}&\frac{\mathrm{dc}(uK_{r-1},k_{r-1})}{\sqrt{k_{r-1}}}\\&\quad=\frac{\sqrt{k_{r-1}}}{2}\left[\frac{\mathrm{dc}(uK_r,k_r)}{\sqrt{k_r}}+\frac{\sqrt{k_r}}{\mathrm{dc}(uK_r,k_r)}\right]\\&\qquad\text{with }k_{r-1}=\frac{2\sqrt{k_r}}{1+k_r}.\end{aligned}\tag{7}$$

Here, $r$ is limited to $r=1,\cdots,R$. This kind of Landen transformation is called descending because from (7), it can readily be seen that if $r$ is increased, the modulus $k_r$ is getting smaller. Under the assumption that the modulus is sufficiently small after $R$ iteration steps, the function $\mathrm{dc}(u,k_R)$ can be approximated by a trigonometric function [8], i.e., $\mathrm{dc}(u,k_R)\approx\sec(u)$ and $K_R\approx\pi/2$.

In order to derive the ascending Landen transformation, the imaginary transformation $\mathrm{dc}(ju,k)=\mathrm{dn}(u,k')$ is applied to (7). This leads to a formula for the elliptic function $dn$:

$$\begin{aligned}&\mathrm{dn}(uK_{r-1},k'_{r-1})\Big/\sqrt{k_{r-1}}\\&\quad=\sqrt{k_{r-1}}\left[\mathrm{dn}(uK_r,k'_r)\Big/\sqrt{k_r}+\sqrt{k_r}\Big/\mathrm{dn}(uK_r,k'_r)\right]/2.\end{aligned}$$

From the particular case $u=K'_{r-1}/K_{r-1}$ of this equation and the identities $\mathrm{dn}(K',k')=k$ and $\mathrm{dn}(K'/2,k')=\sqrt{k}$, the relation $2K'_{r-1}K_r=K_{r-1}K'_r$ between the complete elliptic integrals follows.

If this relation is inserted in the previous equation and the roles of the primed and unprimed quantities are interchanged subsequently, one can find the formula of the ascending Landen transformation:

$$\begin{aligned}&\frac{\mathrm{dn}(2uK_{r-1},k_{r-1})}{\sqrt{k'_{r-1}}}\\&\quad=\frac{\sqrt{k'_{r-1}}}{2}\left[\frac{\mathrm{dn}(uK_r,k_r)}{\sqrt{k'_r}}+\frac{\sqrt{k'_r}}{\mathrm{dn}(uK_r,k_r)}\right]\\&\qquad\text{with }k'_{r-1}=\frac{2\sqrt{k'_r}}{1+k'_r}.\end{aligned}\tag{8}$$

Again, $r$ is restricted to $r=1,\cdots,R$. Now, if the complementary modulus $k'_R$ is sufficiently small after $R$ iteration steps, this means the modulus $k_R$ is close to one, the elliptic function $\mathrm{dn}(u,k_R)$ is approximately equal to a hyperbolic function [2], [8], i.e., $\mathrm{dn}(u,k_R)\approx\mathrm{sech}(u)$, and $K_R\approx\ln(4/k'_R)$.

### B. Poles of the Transfer Function

For the design of an antimetric or a symmetric wave digital filter, the poles of the canonical reflectances $S_1$ and $S_2$ will be determined, which can uniquely be calculated from the poles of the transfer function [7]. The poles of the transfer function $S_{21}(\psi)$, which describes a causal and stable system, are given by the Hurwitz part of the denominator of the product $S_{21}(\psi)S_{21}(-\psi)$ [cf. (2)].

In order to determine the poles of $S_{21}(\psi)S_{21}(-\psi)$, the zeros of the numerator of (2) must be known. Considering $C(\psi)=-C^*(-\psi^*)$ and (5)

$$C(\psi)=\pm1\quad\Longleftrightarrow\quad jC_{\min}\,\mathrm{dc}(nuG,g)=\pm1$$

has to be solved. In order to apply descending Landen transformation,[3] it is useful to divide both sides by $\sqrt{C_{\min}C_{\max}}$, i.e.,

$$C_0 \stackrel{\text{def}}{=} j\,\mathrm{dc}(nuG,\,g)\Big/\sqrt{g} = \pm 1\Big/\sqrt{C_{\min}C_{\max}} \stackrel{\text{def}}{=} \pm\,\Gamma_0. \tag{9}$$

Then, (7) can be reformulated as

$$C_r = \sqrt{g_r}[C_{r+1} - 1/C_{r+1}]/2 = \pm\Gamma_r \tag{10}$$

where $r = 0, \cdots, R-1$. This equation allows the determination of the coefficients $\Gamma_r$, which are significant for the iteration method:

$$\Gamma_{r+1} = \Gamma_r\Big/\sqrt{g_r} + \sqrt{\Gamma_r^2/g_r + 1}\,. \tag{11}$$

If the modulus $g_R$ is sufficiently small after $R$ iteration steps, the chain of Landen's transformation can be terminated, and the elliptic function is approximated by a trigonometric function, i.e., $\Gamma_R \approx \pm j/[\sqrt{g_R}\cos(nu\pi/2)]$. Separating this complex equation into two real equations, it follows that

$$u = (2\mu-1)n \pm j\,2(-1)^{\mu}\mathrm{arsinh}(1(\Gamma_R\sqrt{g_R}(n\pi), \quad \mu \in \mathbb{Z}. \tag{12}$$

The transformation of the frequency variable depends on the value of the modulus $k$.

*Case 1—Filters with Wide Transition Band* ($k \le 1/\sqrt{2}$)*:* Since the modulus $k$ is smaller than the complementary modulus $k'$, it is recommended to transform the frequency variable with the descending Landen transformation. For that purpose, the equivalent frequency is normalized in order to be able to apply (7) immediately:

$$\begin{aligned}\psi_0 &\stackrel{\text{def}}{=} j\,\mathrm{dc}(uK,\,k)/\sqrt{k} = \psi/\sqrt{\varphi_c\varphi_s},\\ \psi_{s-1} &= \sqrt{k_{s-1}}(\psi_s - 1/\psi_s)/2, \quad s = 1,\cdots,S.\end{aligned} \tag{13}$$

Again, if the modulus $k$ is sufficiently small after $S$ iteration steps, the function dc can be approximated by a trigonometric function in accordance with (7). In agreement with (12), we find for $\nu = 1, \cdots, (n+\delta_n)/2$ that

$$\begin{aligned}\psi_{S,\,\nu} &\approx j\Big/\Big(\sqrt{k_S}\cos(u_\nu\pi/2)\Big)\\ u_\nu &= (2\nu-1)/n + 2j\,\mathrm{arsinh}\left(1/\left(\Gamma_R\sqrt{g_R}\right)\right)/(n\pi).\end{aligned} \tag{14}$$

These quantities are tracked back according to (13), where the real parts of the poles have to be negative.

*Case 2—Filters with Narrow Transition Band* ($k \ge 1/\sqrt{2}$)*:* In contrast to the previous case, it is now more efficient to transform the frequency variable by using the ascending Landen transformation (8). For this purpose, the equivalent frequency (5) is expressed by the function dn through the relations $k^2\mathrm{sn}(u,\,k)^2 + \mathrm{dn}(u,\,k)^2 = 1$ and $\mathrm{sn}(u+K,\,k) = \mathrm{cd}(u,\,k)$ [8]. For the transformation, a modified frequency quantity is introduced: $\psi_0 \stackrel{\text{def}}{=} j\,\mathrm{dn}[(u+1)K,\,k]/\sqrt{k'}$. Considering these conventions, (8) can be written as

$$\psi = j\,\varphi_c\Big/\sqrt{1 + k'\psi_0^2}$$

and

$$\psi_{s-1} = \sqrt{k'_{s-1}}(\psi_s - 1/\psi_s)/2, \quad s = 1,\cdots,S. \tag{15}$$

Under the condition that the complementary modulus $k'_S$ is sufficiently small after $S$ transformation steps, the elliptic function dn can be approximated by a hyperbolic function. According to (12), we obtain

$$\begin{aligned}\psi_{S,\,\nu} &\approx \frac{j}{\sqrt{k'_S}\,\cosh[(u_\nu+1)2^{-S}\ln(4/k'_S)]}\\ u_\nu &= \frac{2\nu-1}{n} + \frac{2j}{n\pi}\mathrm{arsinh}\left(\frac{1}{\Gamma_R\sqrt{g_R}}\right)\end{aligned} \tag{16}$$

[3]For filters being used in technical applications, the modulus $g$ is always much smaller than $1/\sqrt{2}$. In this case, it is advised to apply the descending Landen transformation.

with $\nu$ as in (14). The required poles $\psi_\nu$ are evaluated by employing (15), where the real parts have to be negative.

### C. Darlington's Algorithm

In order to compare Darlington's formulae with the formulae in this correspondence, one has to notice that Darlington does not transform the modulus rather than auxiliary quantities that are reciprocals of the roots of the moduli. In his transformation formulae, the amplitude $C_r(u)$ and the $\psi_s(u)$ are considered for the same transformation steps $r = s$. Furthermore, the usual complex frequency variable $p$ of Darlington's paper can be replaced by the equivalent complex frequency $\psi$. Instead of computing the poles in $p$, Darlington uses a further auxiliary quantity $y = -jp$ for his transformation formulae.

Moreover, all transfer functions are assumed to be real analytic, especially if antimetric filters are considered. Then, the modified characteristic function can be written as

$$C_r(u) = j^n\,\mathrm{dc}(nuG_r,\,g_r)/\sqrt{g_r}$$

and

$$\psi_r(u) = j\,\mathrm{dc}(uK_r,\,k_r)/\sqrt{k_r}.$$

This equation describes a bireciprocal characteristic function of an elliptic filter, and in consequence, the relation between the complete elliptic integrals of each iteration step is preserved. After $R = 4$ transformations, we may apply the approximation

$$C_R(u) = \frac{1}{C_R(\hat{u})} \approx \sqrt{g_R}\cos(n\hat{u}\pi/2)/j^n$$

and

$$\psi_R(u) = \frac{1}{\psi_R(\hat{u})} \approx \sqrt{k_R}\cos(\hat{u}\pi/2)/j \tag{17}$$

in order to achieve the poles of the transfer function

$$\begin{aligned}\psi_{R,\,\nu} = &-j\sqrt{k_R}\cos[(2\nu-1)\pi/(2n)\\ &+ j\mathrm{arsinh}(\Gamma_R/\sqrt{g_R})/n]\end{aligned} \tag{18}$$

with index $\nu$ as above [cf. (14)]. This means that the solutions of (17) are computed in the same way as the poles of a Chebyshev filter [3], [4]. If (17) is reformulated as

$$C_R(u) = \frac{\sqrt{g_R}}{2}\left[\frac{e^{jn\hat{u}\pi/2}}{j^n} + \frac{(-1)^{\delta_n}j^n}{e^{jn\hat{u}\pi/2}}\right]$$

and

$$\psi_R(u) = \frac{\sqrt{k_R}}{2}\left(\frac{e^{j\hat{u}\pi/2}}{j} - \frac{j}{e^{j\hat{u}\pi/2}}\right) \tag{19}$$

and the auxiliary quantities $\psi_{R+1} \stackrel{\text{def}}{=} e^{j\hat{u}\pi/2}/j$ and $C_{R+1} \stackrel{\text{def}}{=} \psi_{R+1}^n$ are introduced, we obtain equivalent transformation formulae according to (10) and (13). From the demand $C_{R+1} = \pm j^{\bar{\delta}_n}\Gamma_{R+1}$, one can obtain the $\psi_{R,\,\mu}$ that have the same representation as the poles of a Butterworth filter, i.e.,

$$\psi_{R+1,\,\kappa} = \sqrt[n]{\Gamma_{R+1}}\,e^{j\pi(2\kappa+\bar{\delta}_n)/(2n)} \tag{20}$$

where $\kappa$ are appropriate integers.

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# Efficient Implementations of Pipelined CORDIC Based IIR Digital Filters Using Fast Orthonormal $\mu$-Rotations

Jun Ma, Keshab K. Parhi, Gerben J. Hekstra, and Ed F. Deprettere

***Abstract*—In this correspondence, an efficient implementation of pipelined coordinate rotation digital computer (CORDIC)-based IIR digital filters based on *fast orthonormal $\mu$-rotations* is proposed. Using this method, the Givens rotations are approximated by angles corresponding to orthonormal $\mu$ rotations, which are based on the idea of CORDIC and can perform rotation with minimal number of shift-add operations. We present various methods of construction for such orthonormal $\mu$ rotations. A significant reduction (over 70%) of the number of required shift-add operations is achieved. All types of fast rotations can be implemented as a cascade of only *four* basic types of shift-add stages. These stages can be executed on a modified floating-point CORDIC architecture, making the pipelined filter highly suitable for VLSI implementations.**

***Index Terms*—IIR digital filters, Jacobian matrices, orthogonal transforms, pipelining processing.**

## I. Introduction

Coordinate rotation by digital computer (CORDIC)-based cascade orthogonal IIR digital filters [1]–[4] are digital filters whose internal computations consist of only orthogonal transformations. These filters possess desired properties for VLSI implementations such as regularity, sharp transition band, low sensitivity to finite precision arithmetic, elimination of limit cycle and overflow oscillations, and stability in spite of parameter quantization. Recently, fine-grain pipelined CORDIC-based IIR digital filters were developed. These filters can perform the filtering operations at arbitrarily high sample rates at the cost of linear increase in hardware complexity [5], [6]. The pipelined filter architectures consist of only Givens rotations and a few additions that are suitable for CORDIC-based VLSI implementations [7].

The implementation complexity is mainly determined by the complexity of the rotation evaluation or angle computation. Different approaches for implementing the rotations and modifying the rotations to reduce the complexity have been presented. They include the *CORDIC method* [8], *approximate rotations* [9], [10] *factorized rotations* [11], [12], *factorized approximate rotations* [13], and *orthonormal $\mu$-rotations* [14], [15]. The efficiency of the modified schemes depends on the particular parallel architecture. With respect to VLSI implementation, a CORDIC method is favorable, where CORDIC-like approximate rotations or orthonormal $\mu$-rotations are attractive compared with the use of the exact CORDIC.

This correspondence presents an efficient implementation of pipelined CORDIC-based IIR digital filters based on fast orthonormal $\mu$ rotations. Using this method, the Givens rotations are approximated by angles corresponding to orthonormal $\mu$ rotations, which are based on the idea of CORDIC and share the property that the rotation requires a minimal number of shift–add operations. We present various methods of construction for such orthonormal $\mu$ rotations. A significant reduction (over 70%) of the number of required shift–add operations is achieved. All types of fast rotations can be implemented as a cascade of only four basic types of shift–add stages. These stages can be executed on a modified floating-point CORDIC architecture, making the pipelined filter highly suitable for VLSI implementations.

## II. Pipelined CORDIC-Based Cascade IIR Digital Filters

The CORDIC-based IIR digital filters are developed for the realization of any stable, passive digital rational real transfer function in a cascaded interconnection of orthogonal sections. Each orthogonal section realizes one real zero or a pair of complex conjugate zeros of the transfer function. The cascade implementation leads to low sensitivity to finite word-length truncation in the filter stop band, whereas the orthogonality of the filter guarantees the low sensitivity in the filter passband. Therefore, these filters have good numerical properties over the entire frequency band [4], [5]. A typical fourth-order CORDIC-based IIR digital filter architecture is shown in Fig. 1 [6]. Notice that the critical path in Fig. 1 goes forward and backward through the entire filter structure, which contains seven multiplications and seven additions. The maximum sample rate is limited by the computation time in the feedback loop. In order to further increase the maximum throughput, fine-grain pipelined CORDIC-based IIR digital filter architectures have been developed using the *constrained filter design* and *polyphase decomposition* technique [6]. A three-level pipelined 12th-order CORDIC IIR digital filter architecture adapted from [6] is shown in Fig. 2, which is pipelined at fine-grain level with a linear increase in terms of number of CORDIC units.

In practice, implementations of Givens rotations using exact CORDIC arithmetic can be expensive. For example, using floating point data format with 16-bit mantissa and 5-bit exponent, it will require approximately 20 pairs of shift-add units for one Givens rotation. This is mainly due to the fact that all angles are realized using the same number of $\mu$-rotation stages, which is usually not necessary. In this correspondence, we present Givens rotation implementations using the so-called *fast orthogonal $\mu$-rotations* [15], [16]. Using this

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