Synthesis and Design of Passive Runge-Kutta Methods

Dietrich Fränken and Karlheinz Ochs

Dedicated to Professor Alfred Fettweis on the occasion of his 75th birthday

Abstract Recently, the concept of passive RUNGE-KUTTA methods for numerical integration has been presented. With the fundamental theory thus provided, the goal of this paper is to cover some of the aspects relevant for an actual implementation of the methods derived. For this, procedures for the synthesis of corresponding wave flow diagrams are developed. Some design examples for PDIRK methods are given. Finally, a simulation example is presented in order to verify the functionality of the methods derived.

Keywords Passive integration, Runge-Kutta methods, Wave digital structures

1. Introduction

A large variety of methods for the numerical integration of differential equations exists. It has been shown that the application of principles formerly introduced for so-called wave digital filters (for an overview, see [1]) may be viewed as a numerical integration of electrical network equations by means of specific linear multistep methods [2]. The resulting algorithms possess some advantageous numerical features which are implied by a property called passivity where the latter can be ensured even under finite word-length conditions. The principles may be applied to both linear or non-linear circuits, cf. [2]–[5].

Beside linear multistep methods, RUNGE-KUTTA methods may be used for numerical integration as well. These methods can be found at several places in literature, cf. [6]–[8]. In recent publications, so-called symplectic RUNGE-KUTTA methods have been investigated which try to map energy-conserving properties of HAMILTONian systems onto the numerical algorithms, cf. [9]–[12].

Now, it has been shown how the concept of passivity known for linear multistep methods can be generalized to RUNGE-KUTTA methods [13]. With a suitable formulation of such a method, a characteristic impedance matrix \( Z(\psi) \) was introduced which, for an \( s \)-stage method, is an \( s \times s \) matrix whose entries are rational functions of degree one at most. The class of passive RUNGE-KUTTA methods, i.e., methods with positive characteristic impedance matrices, was investigated. In particular, methods realizable by means of wave digital structures, namely passive diagonal implicit RUNGE-KUTTA (PDIRK) methods, were considered.

In the following, the synthesis of passive RUNGE-KUTTA methods will be dealt with. Network realizations of the characteristic impedance matrices will be derived which allow for a parametric representation of this class of integration methods. For PDIRK methods with up to three stages, corresponding wave digital structures will be presented where the state variables coincide with the sought numerical solutions. Afterwards, a design of PDIRK methods with optimal consistency order will be performed. Finally, simulation results for a non-linear model will verify the functionality of the methods derived.

2. Passive Runge-Kutta methods

Let us consider a single (linear) capacitance which is described by the differential equation

\[
\tau(t) = C \dot{\tilde{v}}(t)
\]

where the dot denotes the time derivative. It has been shown [13] that an \( s \)-stage RUNGE-KUTTA method with node vector \( k \), weighting vector \( g \), and procedure matrix \( D \) applied to this equation may be written in the condensed form

\[
u(k) = e v(k) + T D C^{-1} j(k)
\]

\[
v(k + 1) = v(k) + T g^T C^{-1} j(k)
\]

with \( e := [1, \ldots, 1]^T \) and the vectors \( u := [v_1, \ldots, v_s]^T \) and \( j := [i_1, \ldots, i_s]^T \). Here, \( v_\sigma \) and \( i_\sigma \) denote the stage voltages and stage currents, respectively, and the value \( v(k) \) approximates the exact solution \( \tilde{v}(t_k) \) of the differential equation at equi-distance time instances \( t_k = t_0 + kT \).

2.1 Characteristic impedance matrix

The characteristic impedance matrix corresponding to a RUNGE-KUTTA method is obtained by a steady-state analysis of eqs. (2) and (3) for stage currents of the form

\[
j(k) = J e^{i \phi_k}.
\]

In steady state, the stages voltages may be written in the form \( u(k) = U e^{i \phi} \) and the complex amplitudes \( J \) and \( U \) are related by the (normalized) characteristic impedance matrix. In detail, we write \( U = R e Z(\psi) J \) with the normalization resistance

\[
R_e := T/(2C)
\]
and the equivalent complex frequency
\[
\psi := \tanh(pT/2) = \frac{e^{pT} - 1}{e^{pT} + 1}.
\] (5)

With this, a Runge-Kutta method is called passive if the characteristic impedance matrix \(Z(\psi)\) is a positive (matrix) function [13], i.e., if \(Z(\psi) + Z^*(\psi)\) is positive semi-definite for all \(\psi\) with non-negative real part. It has been argued that, in case of a passive method, the weighting vector must be of the form \(\mathbf{g}^T = \mathbf{r}^T\) with positive value \(r\) and the characteristic impedance matrix may be written in the form
\[
Z(\psi) = \frac{r}{\psi} e e^T + Z' \quad \text{with} \quad Z' = 2D - r e e^T \tag{6}
\]
where \(Z'\) is positive semi-definite.

### 2.2 Parametric representation

At this point, we want to find a parametric representation for the class of passive Runge-Kutta methods with real coefficients. To this end, we synthesize a network corresponding to the impedance matrix \(Z(\psi)\). The basic idea is to decompose the matrix into
\[
Z(\psi) = \frac{r}{\psi} e e^T + Z'_s + Z'_a \tag{7}
\]
where the first term represents the reactive part and the last two terms are the symmetric and skew-symmetric (antisymmetric) part of \(Z'\), respectively:
\[
Z'_s := \frac{1}{2} (Z' + Z'^T), \quad Z'_a := \frac{1}{2} (Z' - Z'^T). \tag{8}
\]
The sum of these particular impedance matrices can immediately be realized as a series interconnection of three \(s\)-ports.

For a synthesis of the reactive part, a one-port with normalized impedance \(r/\psi\), which for obvious reasons is denoted as a capacitance, is connected in parallel to all ports as shown in Fig. 1.

The symmetric part \(Z'_s\) mirrors the dissipative part of the impedance matrix and reflects a reciprocal resistive network. Any such network can be synthesized by means of ideal transformers and resistances [14]. In Fig. 2, canonic realizations for up to three stages are shown. They imply
\[
Z'_s = \begin{bmatrix} r_a & n_{21}r_1 & n_{31}r_1 \\ n_{21}r_1 & r_b & n_{21}n_{31}r_1 - n_{32}r_2 \\ n_{31}r_1 & n_{21}n_{31}r_1 - n_{32}r_2 & r_c \end{bmatrix}, \tag{9}
\]
respectively, with the particular values
\[
r_a = r_1, \quad r_b = r_2 + n_{21}^2r_1 + r_2, \quad r_c = n_{31}^2r_1 + n_{32}^2r_2 + r_3. \tag{10}
\]
The skew-symmetric part \( Z'_a \) represents a lossless \( s \)-port which can be synthesized by a gyrator network. Examples are given in Fig. 3. We get

\[
Z'_a = 0, \quad Z'_a = \begin{bmatrix} 0 & -r_{21} \\ r_{21} & 0 \end{bmatrix}.
\]

With these results and the relationship \( g^T = re^T \), the class of passive RUNGE-KUTTA methods is completely parameterized because the procedure matrix \( D \) can now be determined from eq. (6) and the node vector \( k \) is then given by BUTCHER’s node condition [6]

\[
k = De.
\]  

For our purposes, the values of the circuit elements are suitable parameters in contrast to e.g. the procedure coefficients. This is because the requirement of passivity can simply be fulfilled as the resistances have to be non-negative only. Of course, such a restriction does not apply to both the gyration resistances and the transfer ratios.

3. Synthesis of PDIRK methods

One can check that, for each number of stages, the overall number of electrical parameters in the networks given above matches the number of independent matrix elements in the characteristic impedance matrix \( Z(\psi) \). But, it has been argued [13] that, if a passive RUNGE-KUTTA method is to be realized by means of wave digital structures, the matrix \( Z(1) = 2D \) has to be of lower triangular form, i.e., a passive diagonal implicit RUNGE-KUTTA (PDIRK) method has to be used. Due to this, the gyration resistances in \( Z'_a \) turn out to be fixed as follows:

\[
r_{21} = r + n_{21}r_1, \quad r_{31} = r + n_{31}r_1, \\
r_{32} = r + n_{21}n_{31}r_1 - n_{32}r_2.
\]  

With these restrictions, the procedure matrix of any PDIRK method with up to three stages may in dependence of the normalized circuit parameters be written as

\[
D = \frac{1}{2} \begin{bmatrix} r + r_a & 0 \\ 2r_{21} & r + r_b \end{bmatrix}, \quad D = \frac{1}{2} \begin{bmatrix} r + r_a & 0 & 0 \\ 2r_{21} & r + r_b & 0 \\ 2r_{31} & 2r_{32} & r + r_c \end{bmatrix}.
\]  

3.1 Some principles of wave digital structures

In order to obtain a wave digital structure from a given circuit, so-called wave quantities are introduced. To this end, a positive port resistance is assigned to each port of the multiport under consideration. Let \( R \) denote the diagonal matrix of the \( s \) normalized port resistances and \( G \) be its inverse. Then, the (voltage) wave quantities are defined by

\[
a := u + R R_i \quad \text{and} \quad b := u - R R_i
\]  

With these definitions, an \( s \)-port with normalized impedance matrix \( Z(\psi) \) may alternatively be characterized by its scattering matrix \( S(\psi) \) which relates the complex amplitudes of incident and reflected waves in steady state and is given by

\[
S(\psi) = (Z(\psi) - R)(Z(\psi) + R)^{-1} \iff Z(\psi) = (1 + S(\psi))(1 - S(\psi))^{-1} R
\]  

with the identity matrix \( I \).

Now, the equations imposed by both the network elements as well as the Kirchhoff laws may be rewritten in dependence on the wave quantities introduced. With suitably chosen port resistances, standard wave digital elements follow: a resistive voltage source is identical to a wave source, a capacitance becomes a simple time delay, series and parallel connections are realized by so-called adaptors, etc. [1]. In many cases, these elements can be port-wise interconnected in order to obtain a suitable wave digital structure where passivity of the overall structure is implied by passivity of the single elements and can easily be ensured even under finite word-length conditions.

3.2 Wave digital structures for PDIRK methods

While the realization of the characteristic impedance of a 1-stage RUNGE-KUTTA method basically is just a series connection of a capacitance and a resistance and can thus be synthesized by standard procedures, we find that the latter is not true for methods with two or more stages. In order to specify the underlying problem, note that the circuit represented by Figs. 1–3 in particular contains gyrators and hence is non-reciprocal in general. The same statement still holds if the capacitance and all resistances are extracted from the circuit and the remaining non-energetic multiport is considered. Consequently, a canonic synthesis procedure cannot be based on standard procedures known for the synthesis of connection networks (cf. [1], [15]), as those would immediately lead to delay-free directed loops. Hence, an alternate approach has to be adopted here.

First, let us introduce a permutation type \( s \times s \) matrix

\[
P(e^{pT}) := \begin{bmatrix} 0^T & e^{-pT} \\ 1 & 0 \end{bmatrix}.
\]  

Furthermore, let \( L \) denote the strict lower triangular \( s \times s \) matrix where all elements below the main diagonal equal

\[1\] This is true unless methods with diagonal procedure matrices are considered. In this case, the gyrators degenerate to short circuits.
one. By inspection, we verify that the matrices thus defined fulfill the condition

\[(1 e^{pT} + L e^{pT} + L^T) (1 - P(e^{pT})) = (e^{pT} - 1) 1\]

and hence we have

\[\begin{align*}
(1 + P(e^{pT})) (1 - P(e^{pT}))^{-1} &= 2(1 - P(e^{pT}))^{-1} - 1 \\
&= e^{pT} + (1 + L + L^T) - 1 \\
&= e^{pT} + 1 (1 + L + L^T) - (L - L^T).
\end{align*}\]

(18)

Now, consider an \(s\)-port whose scattering matrix with respect to equal normalized port resistances \(r\) is given by \(P(e^{pT})\) and which thus represents a circulator structure as given in Fig. 4. Due to the relationship just derived and with \(1 + L + L^T = ee^T\), the normalized impedance matrix of this \(s\)-port is given by

\[Z_p(\psi) = r \frac{e}{\psi} e^T + r (L - L^T).\]

(19)

With this result, it seems to be advantageous to replace the decomposition (7) by

\[Z(\psi) = Z_p(\psi) + Z_q\]

(20)

Fig. 4. Realization of the impedance matrix \(Z_p(e^{pT})\).

Fig. 5. Wave digital structures corresponding to the characteristic impedances of 1-stage and 2-stage RUNGE-KUTTA methods.

With these results, wave digital structures for an implementation of 1-stage and 2-stage RUNGE-KUTTA methods are readily deduced as shown in Fig. 5. They contain, in particular, so-called series adaptors with

\[Z_q = Z' - r (L - L^T) = 2D - r1 - 2rL.\]

(21)

While the wave digital realization of the \(s\)-port with normalized impedance matrix \(Z_p(\psi)\) is immediately apparent from eq. (17), it still remains to find a structure for the impedance matrix \(Z_q\) or, equivalently, for the corresponding scattering matrix \(S_q\). In order to avoid delay-free directed loops later on, the matrix \(S_q\) must be of strict lower triangular form, and the sought structure must not contain any directed signal path from the input at port \(\varrho\) to the output at port \(\sigma\) for \(\varrho \geq \sigma\). In particular, the accessible ports must be reflection-free and hence the port-resistances have to be chosen equal to the diagonal elements of \(Z_q\) which are just the values \(r_a, r_b, a n d r_c\) defined in eq. (10). With these choices, we get

\[S_q = 0, S_q = \begin{bmatrix} 0 & 0 \\ n_{21} & 0 \end{bmatrix},\]

(22)

respectively.

From these results, wave digital structures for an implementation of 1-stage and 2-stage RUNGE-KUTTA methods are readily deduced as shown in Fig. 5. They contain, in particular, so-called series adaptors with
reflection-free ports, cf. [1], a corresponding signal flow diagram is given in Fig. 6. Note that the sign inverters had to be introduced in Fig. 5 in order to account for the voltage orientations in a series connection.

To the knowledge of the authors, a solution to the synthesis problem applicable for arbitrary numbers of stages has not yet been found. Nevertheless, a solution for 3-stage methods will be given in the following. We can observe that, in this case, the scattering matrix can be written in the form

\[
S_q = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
n_{21} \gamma_{21} (n_{32} + \delta_{32}) & \delta_{32} - \gamma_{21} (n_{32} + \delta_{32}) & 0
\end{bmatrix}
\]  

(23)

if we introduce the particular values

\[
\gamma_{21} = \frac{r_2}{r_b} = \frac{r_2}{n_{21} r_1 + r_2} \quad \text{and} \quad \delta_{32} = \frac{n_{31}}{n_{21}}.
\]  

(24)

Hence, a wave digital realization of a 3-stage RUNGE-KUTTA method looks as shown in Fig. 7. It should be emphasized at this point that the structure used for synthesizing \(S_q\) does indeed fit into the wave digital concept. In fact, it turns out to be identical with the structure shown in Fig. 8 if all signal paths bearing zero signals or signals of no further interest are omitted therein. Hence, passivity even under finite word-length conditions can be easily ensured by measures standard in context with wave digital filtering. This motivates the usage of this structure with four multipliers rather than a direct implementation with three multipliers as suggested by the non-zero elements in \(S_q\), because for the latter alternative passivity is hard to ensure.

3.3 State variables and wave quantities

For a PDIRK method which is synthesized as mentioned above, the value stored in the delay element turns out to be identical to the sought solution of the numerical integration method. In other words, the state variable \(b_C\) of the discrete-time system defined by the wave flow diagram is just \(v\).

In order to prove this, we start with the observation that the currents \(i_C\) in Figs. 1 and 4 are in fact identical and that thus the same is true for the voltages \(v_C\). While it is obvious from Fig. 1 that \(e^T J = i_C\) holds the same may be
deduced for Fig. 4 from eq. (17) via

\[ 2rR_C \sum_{\sigma=1}^{s} \tilde{i}_\sigma = \sum_{\sigma=1}^{s} (a_{p\sigma} - b_{p\sigma}) = a_{p1} - b_{p1} = a_C - b_C = 2rR_C^2. \]  

(25)

Now, take a look at eq. (2) which in particular implies

\[ v_\sigma = v_C + \frac{T}{C} d^T \sigma j \]  

(26)

where \( d^T \sigma \) denotes the \( \sigma \)-th row of the procedure matrix \( D \). On the other hand, for the series connection of a reactive and a dynamic-free \( s \)-port as characterized by eq. (6), we may also write

\[ v_\sigma = v_C + \frac{T}{C} \left[ 2d^T \sigma - re^T \right] j. \]  

(27)

With \( e^T j = i_C \), the identity

\[ v = v_C - \frac{rT}{2C} i_C = b_C \]  

(28)

readily follows.

From this result, it becomes obvious that the determination of the initial states for a PDIRK simulation of an electrical circuit is a trivial task.

4. Design of PDIRK methods

In most cases, the design of a RUNGE-KUTTA method (i.e. a selection of appropriate values for the free coefficients) with prescribed consistency order and additional desired properties turns out to be a nontrivial problem. But, with the values of the circuit elements as design parameters, the requirement of passivity can simply be taken into account during the design process.

Starting from the well-known consistency conditions for RUNGE-KUTTA methods, it has been shown [13] that the maximum attainable consistency order of a PDIRK method in general cannot exceed four and is bounded by \( s + 1 \) for an \( s \)-stage method with \( s \leq 3 \). In this section, we consider the relevant equations, i.e. BUTCHER’S node condition (12), the consistency conditions, and the relationship \( g^T e = r e^T \), all written in dependence of the circuit parameters.

Any method with arbitrary stage number \( s \) and consistency order \( q \geq 1 \) must fulfill \( g^T e = 1 \) which immediately implies

\[ r = \frac{1}{s}. \]  

(29)

For a consistency order \( q \geq 2 \), in addition \( g^T k = 1/2 \) has to hold. With this, we must have

\[ r_1 = 0 \]  

(30)

for a one-stage method with consistency order two and thus this optimum is just reached by the well-known GAUSS method.

For \( s = 2 \), the very same condition implies

\[ r_1(n_{21} + 1)^2 + r_2 = 0. \]  

(31)

Due to passivity, we hence must have \( r_2 = 0 \) while the two cases \( r_1 = 0 \) and \( n_{21} = -1 \) are to be dealt with separately. Now, the consistency conditions for \( q \geq 3 \) read \( g^T Dk = 1/6 \) and \( g^T k^2 = 1/3 \). While they are always violated for \( r_1 = 0 \), both conditions lead to

\[ \left( r_1 - \frac{1}{2} \right)^2 = \frac{1}{3} \]  

(32)

for \( n_{21} = -1 \). Finally, passivity may again be exploited to find the unique solution

\[ r_1 = \frac{1}{2} \pm \frac{1}{\sqrt{3}}, \quad r_2 = 0, \quad n_{21} = -1. \]  

(33)

In order to find a three-stage PDIRK method with consistency order four, a set of eight multi-variable polynomial equations has to be solved. One may here make use of the so-called BUCHBERGER algorithm [16] which generates a GRÖBNER basis [17] for solving this set of equations where, similar to the two-stage case, the additional inequalities implied by the demand for passivity can be used to reduce the computational complexity. With help of a computer algebra program, we finally get the (unique) solution

\[ r_1 = \frac{2}{3} + \frac{1}{\sqrt{2}}, \quad r_2 = r_3 = 0, \quad n_{21} = -2, \quad n_{31} = 1. \]  

(34)

As, with \( r_2 = 0 \), the adjacent ideal transformer is short-circuited on both ports, the corresponding turns ratio \( n_{32} \) may be chosen arbitrarily. In Fig. 7, the latter is apparent from the fact that the coefficient \( r_{21} \) becomes zero in this limiting case. With \( n_{21} = -2 \) and \( \delta_{32} = -1/2 \), the structure becomes rather simple.

The circuit parameters for PDIRK methods with optimal consistency order have thus been found. The corresponding procedure coefficients are identical to those published before [13], but their derivation turned out to be simpler.

5. An example

The functionality of the methods derived in the previous sections has been tested by the authors for a variety of linear and non-linear electrical circuits. In the following, a simple example will be presented.

Consider the electrical network shown in Fig. 9 which in particular contains an ideal diode described by

\[ \tilde{v}_D = 0 \text{ for } \tilde{i}_D \geq 0 \quad \text{and} \quad \tilde{i}_D = 0 \text{ for } \tilde{v}_D \leq 0. \]  

(35)
The corresponding differential equation reads
\[ \ddot{v} + RC\dot{v} = 0 \quad \text{for} \quad \ddot{v} \geq \ddot{\ddot{x}} \]  \hfill (36)
and
\[ \ddot{v} \left[ 1 + \frac{R_x}{R} \right] + R_x C \dot{v} = \ddot{\ddot{x}} \quad \text{for} \quad \ddot{v} \leq \ddot{\ddot{x}}. \]  \hfill (37)

Now, if the input signal is chosen to be $T_x$-periodic with
\[ \ddot{x}(t) = \begin{cases} \ddot{x} > 0 & \text{for } 0 < t < T_x/2 \\ 0 & \text{for } T_x/2 < t < T_x \end{cases}, \]  \hfill (38)
the piecewise linear differential equation can easily be solved in closed form. With this, we will be able to determine the error produced by the numerical integration algorithm.

From the circuit in Fig. 9, the corresponding wave digital structure shown in Fig. 10 can be obtained. Here, the normalized port resistances have to be chosen as indicated while a simple analysis shows that the diode is described by
\[ b_D = -|a_D| \]  \hfill (39)

independently of the corresponding port resistance [3].

Of course, the structure given here results from an application of the standard trapezoidal rule which possesses the maximum consistency order attainable with a passive linear multistep method, namely two [18]. For the simulation with an $s$-stage PDIRK method, the time delay has to be replaced by the corresponding structure of Figs. 5 or 7, respectively. This is then to be terminated

by $s$ wave flow diagrams which are of identical structure but may differ in their coefficient values because, in particular, the values of the adptor coefficients depend on the port resistances of the accessible ports and thus on the parameters of the chosen PDIRK method. In order to visualize the approach presented in this paper, the overall wave digital structure resulting for a 2-stage method is shown in Fig. 11. Here, the two dynamic-free parts of the structure perform the evaluation of the implicit non-linear equations
\[ C^{-1}f_a(k) = f(v_a(k), \ddot{x}(t_k + k\sigma T)) \]  \hfill (40)
in the explicit form
\[ a_\sigma(k) = g(b_\sigma(k), \ddot{x}(t_k + k\sigma T)). \]  \hfill (41)

The circuit has been simulated for the parameter values $R = R_x = 1\, \Omega$, $C = 1\, \text{F}$, $\ddot{x} = 1\, \text{V}$, $T_x = 1\, \text{s}$, and $v_0(t) = \dddot{v}(0) = 0.8\, \text{V}$. In Fig. 12, the simulation results for different PDIRK methods are displayed. Because the implementation of an $s$-stage PDIRK method in particular requires the evaluation of $s$ algebraic equations, the simulation sampling rates $\sigma = 1/T$ were chosen as indicated in order to have comparable overall implementation efforts. It can be seen that the usage of PDIRK methods with higher consistency orders does indeed significantly increase the numerical accuracy. In fact, further simulations have shown that, for the example under consideration, the sampling rate has to be increased as much as up to 2500 Hz if the error for the one-stage method is to be of the same magnitude as that for the three-stage method. In other words, for a simulation of the circuit in Fig. 9 with the chosen parameter values, the overall implementation effort of the method which is identical to the GAUSS method and thus comparable to the trapezoidal rule is approximately eight times higher than that of the 3-stage method derived in this paper. A netto reduction of the implementation effort by about a factor four is still achieved, when the 2-stage method with consistency order three is used instead of the trapezoidal rule. The latter alternative seems to be especially attractive as the two dynamic-free wave flow diagrams are completely identical including the values of the adaptor coefficients as a consequence of eqs. (10) and (33).

6. Conclusions

In this paper, synthesis and design of passive numerical integration methods have been dealt with. This class of integration methods is of particular interest because with passivity a lot of desirable numerical stability properties are automatically ensured.

For a passive RUNGE-KUTTA method with up to three stages, a canonion normalized reference circuit has been given. This kind of description not only gives the engineer an intuitive understanding of the underly-
Fig. 11. Wave digital structure corresponding to the non-linear electrical circuit shown in Fig. 9 for a 2-stage PDIRK method. The input signals are given by \( x_\text{in}(k) = \tilde{x}(t_k + k\, T) \) where the nodes \( k_{\text{in}} \) are determined by BUTCHER’s node condition (12).

Fig. 12. Simulation of the electrical circuit given in Fig. 9: Excitation, exact solution, and simulation errors for different PDIRK methods with comparable implementation efforts.

For a realization by means of wave digital structures, passive diagonal implicit RUNGE-KUTTA methods are to be employed. Wave digital structures have been presented which allow for an implementation of these methods, again for up to three stages. Design examples for methods with maximum attainable consistency order have been given.

A simulation example has shown that, by choosing a suitable class of RUNGE-KUTTA methods, we were indeed able to increase the accuracy in comparison to the formerly known passive linear multistep methods. This has been done without sacrificing passivity and the advantageous properties induced by the latter.

Acknowledgement

The authors wish to thank Professor Klaus Meerkötter for many valuable hints and fruitful discussions on the topic of this paper.

References


Dietrich Fränken was born in Berlin, Germany in 1966. He received the the Diploma in Electrical Engineering from the Ruhr-Universität Bochum, Germany in 1991 and the Dr.-Ing. degree from the University of Paderborn, Germany in 1997, respectively. Research interests include system theory and digital signal processing.

Karlheinz Ochs was born in Lingen, Germany in 1968. He received the Diploma in Electrical Engineering from the University of Paderborn, Germany, in 1996. Currently, he is a Ph.D. candidate at the University of Paderborn, Germany. Research interests include system theory and digital signal processing.