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Improving wave digital simulation by extrapolation techniques

Dietrich Fränken and Karlheinz Ochs

Abstract **Wave digital simulation principles are known to exhibit excellent numerical stability properties due to their inherent passivity. Yet, the achievable accuracy may not suffice in all applications. In this paper, it will be discussed how extrapolation techniques known from numerical mathematics can be merged with the wave digital concept. The proposed technique can be applied to both ordinary and partial differential equations. As two examples will show, high numerical accuracy can be obtained this way without sacrifycing passivity and the advantageous properties induced by the latter.**

Keywords **Wave digital simulation, Extrapolation, Passive integration methods, Linear multistep methods**

1. Introduction

A large variety of methods for the numerical integration of differential equations exists. It has been shown that the application of principles formerly introduced for so-called wave digital filters (for an overview, see [1]) may be viewed as a numerical integration of KIRCHHOFF network equations by means of specific linear multistep methods [2]. The resulting algorithms possess some advantageous numerical features which are implied by a property called passivity where the latter can be ensured even under finite word-length conditions. The principles may be applied to both linear or non-linear circuits being described by ordinary or partial differential equations, cf. [2]-[9].

Unfortunately, the numerical accuracy achievable by a mere usage of passive linear multistep methods may not suffice in combination with a justifiable computational effort. Consequently, RUNGE-KUTTA methods have been investigated in recent publications in order to obtain an increased numerical accuracy without sacrificing passivity and the stability properties induced herewith. Appropriate methods have been introduced and shown to fulfill the aforementioned requirements at least for ordinary differential equations [10, 11].

In this paper, a different idea will be followed. After a short recapitulation of basic terms and definitions, a frequency-based approach will be discussed. Although an increased accuracy here (for linear multistep methods) comes along with a loss of passivity, the approach turns out to be related to so-called extrapolation methods known from numerical mathematics. The latter can, as we will show by means of two simulation examples, indeed successfully be used for increasing the numerical accuracy of simulations based on the wave digital concept. This is of particular interest in context with partial differential equations where the application of RUNGE-KUTTA methods often is not possible offhand.

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2. Linear multistep methods

Linear multistep methods are widely used for the numerical integration of ordinary as well as of partial differential equations. In this section, some basic definitions and facts are shortly recalled.

2.1 Method, local and global error, consistency order

For the time being, let us consider a KIRCHHOFF network which, in addition to a voltage source and a single capacitance, only contains non-dynamic and source-free elements. The voltage $\tilde{v}$ and the current $\tilde{\imath}$ at the capacitance are described by the differential equation

$$\dot{\tilde{v}}(t) = C^{-1}\tilde{\imath}(t) \quad \text{with} \quad \tilde{v}(t_0) = v_0\,, \tag{1a}$$

while the other elements are assumed to impose a (possibly nonlinear) algebraic equation of the form

$$C^{-1}\tilde{\imath}(t) = f(\tilde{v}(t), x(t)) \tag{1b}$$

where x is the source voltage.

Now, a linear multistep method numerically solves the differential equation (1a) by replacing it with

$$v(t_k) = -\sum_{\sigma=1}^{s} \alpha_\sigma v(t_{k-\sigma}) + TC^{-1}\sum_{\sigma=0}^{s} \beta_\sigma i(t_{k-\sigma}) \tag{2}$$

where α_σ, β_σ are suitably chosen parameters, T is the step-size or sampling period, and t_k is defined by

$$t_k := t_0 + kT \tag{3}$$

with $k \in \mathbb{Z}$. In numerical mathematics, the so-called local error is widely used as a measure of quality of numerical

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integration methods [12]. For a linear multistep method, this error is defined by

$$\varepsilon(t_k) := \tilde{v}(t_k) + \sum_{\sigma=1}^{s} \alpha_\sigma \tilde{v}(t_{k-\sigma}) - T\beta_\sigma C^{-1}\tilde{\imath}(t_{k-\sigma})\,. \quad (4)$$

A linear multistep method is said to be of consistency order q if the local error obeys the equation

$$\varepsilon(t_k) = \mathcal{O}\left(T^{q+1}\right) \quad (5)$$

where the LANDAU order symbol $\mathcal{O}$ has been adopted for the purpose of a compact notation. It can be shown that, under some mild restrictions, the global error $v(t) - \tilde{v}(t)$ for any fixed $t > t_0$ is of the form

$$v(t) - \tilde{v}(t) = \mathcal{O}(T^q) \quad (6)$$

for a method of order q [13, 14].

2.2 Characteristic impedance and passive methods

With respect to the linear difference eq. (2), it is common practice to consider the steady state at some complex frequency p where v and i are assumed to be of the form $v(t_k) := V\mathrm{e}^{pt_k}$ and $i(t_k) := I\mathrm{e}^{pt_k}$, respectively [2, 15, 16]. In this case, the linear difference equation yields $V = R\hat{Z}\left(\mathrm{e}^{pT}\right)I$ where

$$R := \frac{T}{2C} \quad (7)$$

is a normalization resistance and

$$\hat{Z}\left(\mathrm{e}^{pT}\right) := 2\sum_{\sigma=0}^{s} \beta_\sigma \mathrm{e}^{\sigma pT} \Big/ \sum_{\sigma=0}^{s} \alpha_\sigma \mathrm{e}^{\sigma pT} \quad (8)$$

with $\alpha_0 := 1$ denotes the so-called (normalized) characteristic impedance $\hat{Z}$ which is assumed to be of irreducible form. In the sequel, it will be useful to write the characteristic impedance according to

$$Z(\psi) := \hat{Z}\left(\frac{1+\psi}{1-\psi}\right) \quad (9)$$

where ψ is the equivalent complex frequency

$$\psi := \tanh(pT/2) = \frac{\mathrm{e}^{pT}-1}{\mathrm{e}^{pT}+1}\,. \quad (10)$$

As can be shown, a linear multistep method is of consistency order q if

$$\hat{Y}\left(\mathrm{e}^{pT}\right) - \frac{pT}{2} = \hat{\delta}[pT]^{q+1} + \mathcal{O}\left([pT]^{q+2}\right) \quad (11)$$

holds for the characteristic admittance $\hat{Y} := \hat{Z}^{-1}$ with some non-zero local error constant $\hat{\delta}$. Replacing the frequency variable p by ψ, we have

$$Y(\psi) - \operatorname{artanh}(\psi) = \delta\psi^{q+1} + \mathcal{O}\left(\psi^{q+2}\right) \quad (12)$$

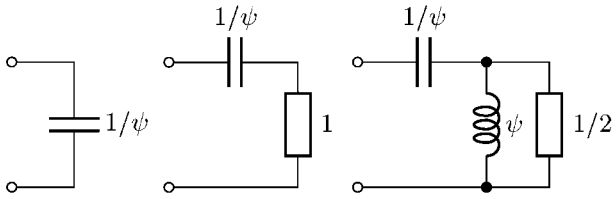

Fig. 1. Characteristic impedances for some passive linear multistep methods. From left to right: trapezoidal rule, implicit EULER and GEAR's procedure.

with a modified error constant δ.

For a wave digital simulation, so-called passive methods are of particular importance. Here, a linear multistep method is called passive if the corresponding characteristic impedance is a positive function [10], i. e., if the implication

$$\Re\psi > 0 \quad \Rightarrow \quad \Re Z(\psi) \geq 0 \quad (13)$$

holds. Characteristic impedances for some passive linear multistep methods are shown in Fig. 1.

3. A frequency-based approach

In this section, a frequency-based approach to improving the numerical accuracy of linear multistep methods will be discussed. It is based on the idea of superimposing characteristic admittances according to versions of one and the same linear multistep method used with different step-sizes.

3.1 Interpretation of changing the step-size

First, we will focus on the effects of altering the step-size. To be more precise, we take a basis step-size T into consideration and relate it to a new step-size nT, where n is a positive integer. In the steady state for some complex frequency p, this results in a transition from ψ to

$$\psi_n := \tanh(npT/2) = \frac{\mathrm{e}^{npT}-1}{\mathrm{e}^{npT}+1}\,. \quad (14)$$

In view of the equation

$$z := \mathrm{e}^{pT} = \frac{1+\psi}{1-\psi}\,, \quad (15)$$

we may also write

$$\psi_n := \frac{z^n-1}{z^n+1} = \frac{[1+\psi]^n - [1-\psi]^n}{[1+\psi]^n + [1-\psi]^n}\,. \quad (16)$$

Hence, the change in step-size can be viewed as a (rational) frequency transformation where, for a linear multistep method, we have instead of $RZ(\psi)$ the new

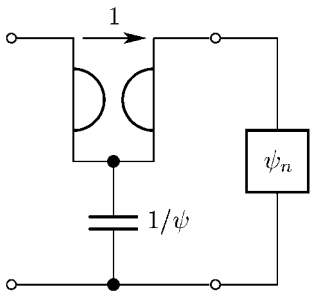

Fig. 2. Concretely lossless KIRCHHOFF network with (normalized) impedance ψ_{n+1}.

(non-normalized) characteristic impedance $R_n Z(\psi_n)$ with $R_n := nT/[2C] = nR$.

In order to achieve a network-theoretical interpretation of this frequency transformation, it is useful to introduce the recurrence formula

$$z^{n+1} = z^n z \quad \Longleftrightarrow \quad \psi_{n+1} = \frac{\psi_n + \psi}{1 + \psi_n \psi} \tag{17}$$

with $\psi_1 = \psi$. Accordingly, the recurrence can be visualized by the network shown in Fig. 2. From this, it is obvious that with ψ_n also ψ_{n+1} represents a lossless network. Consequently, ψ_{n+1} always is a so-called FOSTER function, i. e., it is an odd positive function. Formally, this can be shown e. g. by noting that with ψ_n also ψ_{n+1} obviously is odd and that in (13) the requirement $\Re\psi > 0$ is equivalent to $|z| > 1$.

3.2 Improving the accuracy

At this point, let us examine eq. (11) for two different step-sizes $n_1 T$ and $n_2 T$:

$$\frac{\hat{Y}\left(\mathrm{e}^{n_1 pT}\right)}{n_1} = \frac{pT}{2} + n_1^q \hat{\delta} [pT]^{q+1} + \mathcal{O}\left([pT]^{q+2}\right) \tag{18}$$

and

$$\frac{\hat{Y}\left(\mathrm{e}^{n_2 pT}\right)}{n_2} = \frac{pT}{2} + n_2^q \hat{\delta} [pT]^{q+1} + \mathcal{O}\left([pT]^{q+2}\right). \tag{19}$$

Now, form a weighted sum of the two expressions on the left hand side in order to obtain an increased consistency order of at least $q+1$. To this end, weights g_1 and g_2 have to be chosen such that

$$g_1 \frac{\hat{Y}\left(\mathrm{e}^{n_1 pT}\right)}{n_1} + g_2 \frac{\hat{Y}\left(\mathrm{e}^{n_2 pT}\right)}{n_2} = \frac{pT}{2} + \mathcal{O}\left([pT]^{q+2}\right)$$

holds. This requirement leads to a set of linear equations from which the weights can be determined.

Alternatively, the result can be obtained by multiplying both eq. (19) with n_1^q and eq. (18) with n_2^q, respectively, and subsequently dividing the difference of the resulting equations by $n_1^q - n_2^q$ afterwards:

$$\frac{\frac{\hat{Y}\left(\mathrm{e}^{n_2 pT}\right)}{n_2} n_1^q - \frac{\hat{Y}\left(\mathrm{e}^{n_1 pT}\right)}{n_1} n_2^q}{n_1^q - n_2^q} = \frac{pT}{2} + \mathcal{O}\left([pT]^{q+2}\right).$$

Reformulating this equation we thus can write the (normalized) characteristic impedance $Y_\mathrm{e}(\psi)$ of the new method in dependence of the equivalent frequency variable ψ according to

$$Y_\mathrm{e}(\psi) = \frac{Y\left(\psi_{n_2}\right)}{n_2} + \frac{\frac{Y(\psi_{n_2})}{n_2} - \frac{Y(\psi_{n_1})}{n_1}}{[n_1/n_2]^q - 1}. \tag{20}$$

Here,

$$Y_\mathrm{e}(\psi) = \operatorname{artanh}(\psi) + \mathcal{O}\left(\psi^{q+2}\right) \tag{21}$$

holds. Hence, the new method has an improved consistency order of at least $q+1$. An even better enhancement can be achieved if one starts from $Y(\psi) = \psi$ corresponding to the trapezoidal rule. In this case, the consistency order is improved from 2 to 4.

For the more general case, let us now consider m different step-sizes $n_1 T$, $n_2 T$, $n_3 T$ and so forth and build the linear combination

$$Y_\mathrm{e}(\psi) := \sum_{\mu=1}^{m} g_\mu \frac{Y(\psi_{n_\mu})}{n_\mu}. \tag{22}$$

Again, one either may obtain Y_e according to eq. (22) where the coefficients g_μ are determined by a set of linear equations or one may follow the line of arguments leading to the expression (20). For reasons of further presentation, we here choose the second alternative and start from the m terms

$$Y_{\mu,1} = \frac{pT}{2} + \sum_{\lambda=1}^{m-1} \hat{\delta}_\lambda \Delta_{\mu,\lambda}^{[1]} [pT]^{q+\lambda} + \mathcal{O}\left([pT]^{q+m}\right) \tag{23a}$$

with

$$Y_{\mu,1} := \frac{\hat{Y}\left(\mathrm{e}^{n_\mu pT}\right)}{n_\mu} \quad \text{and} \quad \Delta_{\mu,\lambda}^{[1]} := n_\mu^{q+\lambda-1} \tag{23b}$$

for $\mu = 1, \ldots, m$ and $\lambda = 1, \ldots, m-1$. With a close examination, ones finds that the leading error terms may be eliminated by succesively calculating

$$Y_{\mu,\nu+1} = Y_{\mu,\nu} + \frac{Y_{\mu,\nu} - Y_{\mu-1,\nu}}{\left[\Delta_{\mu-1,\nu}^{[\nu]} \,/\, \Delta_{\mu,\nu}^{[\nu]}\right] - 1} \tag{24a}$$

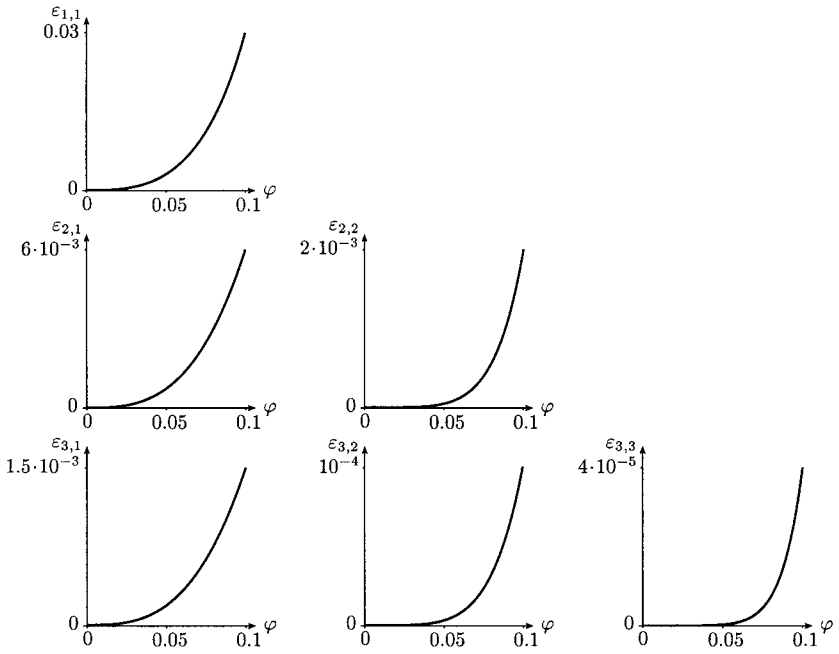

Fig. 3. Deviation $\varepsilon_{\mu,\nu}(\varphi) := |Y_{\mu,\nu}(j\varphi) - \mathrm{j}\arctan(\varphi)|$ for the trapezoidal rule used with the step-sizes $4T$, $2T$, T.

for $\nu = 1, \ldots, m-1$ and $\mu = \nu+1, \ldots, m$ as well as

$$\Delta_{\mu,\lambda}^{[\nu+1]} = \Delta_{\mu,\lambda}^{[\nu]} + \frac{\Delta_{\mu,\lambda}^{[\nu]} - \Delta_{\mu-1,\lambda}^{[\nu]}}{\left[\Delta_{\mu-1,\nu}^{[\nu]} \,/\, \Delta_{\mu,\nu}^{[\nu]}\right] - 1} \tag{24b}$$

for $\lambda = \nu+1, \ldots, m-1$. With transition to the equivalent frequency variable ψ, one finally obtains

$$Y_{\mathrm{e}}(\psi) = Y_{mm} = \operatorname{artanh}(\psi) + \mathcal{O}(\psi^{q+m}), \tag{25}$$

i. e., the consistency order has been improved by $m-1$ at least [12].

For order $q = 1$, it can be shown that eqs. (24) take on the simplified form

$$Y_{\mu,\nu+1} = Y_{\mu,\nu} + \frac{Y_{\mu,\nu} - Y_{\mu-1,\nu}}{\left[n_{\mu-\nu} \,/\, n_{\mu}\right] - 1}\,. \tag{26}$$

Consequently, for methods like the trapezoidal rule being of consistency order two where only even powers of pT occur in (23a), one may use

$$Y_{\mu,\nu+1} = Y_{\mu,\nu} + \frac{Y_{\mu,\nu} - Y_{\mu-1,\nu}}{\left[n_{\mu-\nu} \,/\, n_{\mu}\right]^2 - 1} \tag{27}$$

instead increasing the consistency order by two within each step. Figure 3 visualizes the improvement in accuracy achievable this way.

3.3 Achievable numerical accuracy

While the frequency-based approach can be used to construct linear multistep methods of arbitrary consistency order, this is no longer true when it comes to using passive methods. Clearly, $Y_{\mathrm{e}}(\psi)$ again represents a linear multistep method and it is known that the order of a passive linear multistep method can never exceed 2 (the so-called DAHLQUIST's second barrier [17]).

We can verify this statement by starting from an irreducible characteristic admittance $Y_{\mathrm{e}}(\psi)$ representing a passive method with a consistency order of at least 2:

$$Y_{\mathrm{e}}(\psi) - \operatorname{artanh}(\psi) = \delta\psi^3 + \mathcal{O}\left(\psi^4\right). \tag{28}$$

Now, the characteristic impedance $Z_{\mathrm{e}}(\psi)$ of a method with order $q \geq 1$ must have a simple pole in the origin with residue 1 ([10], also compare with Fig. 1):

$$\lim_{\psi\to 0} \psi Z_{\mathrm{e}}(\psi) = 1\,. \tag{29}$$

Due to this, a multiplication of both sides in eq. (28) with $Z_{\mathrm{e}}(\psi)/\operatorname{artanh}(\psi)$ yields

$$Z_{\mathrm{e}}(\psi) - \frac{1}{\operatorname{artanh}(\psi)} = -\delta\psi + \mathcal{O}\left(\psi^2\right). \tag{30}$$

Replacing $Z_{\mathrm{e}}(\psi)$ by $1/\psi$ and thus δ by $-1/3$, we also have

$$\frac{1}{\psi} - \frac{1}{\operatorname{artanh}(\psi)} = \frac{\psi}{3} + \mathcal{O}\left(\psi^2\right). \tag{31}$$

The subtraction of the two previous equations yields

$$\Delta(\psi) := Z_{\mathrm{e}}(\psi) - \frac{1}{\psi} = \left[-\delta - \frac{1}{3}\right]\psi + \mathcal{O}\left(\psi^2\right) \tag{32}$$

where $\Delta(\psi)$ must again be a positive function because this property remains unaltered by the extraction of a pole located on the imaginary axis [18]. But, $\Delta(\psi)$ being positive immediately implies $\delta \leq -1/3$ which confirms that δ cannot vanish and thus the order cannot be 3 or higher.

4. Extrapolation methods

The frequency-based approach presented in the previous section was inspired by (and is, to a certain extent, related to some) so-called extrapolation methods which are known to often provide numerical solutions of differential equations in an effective way. First extrapolation methods originate to RICHARDSON [19, 20] and a variety of different methods can be found in numerical literature. The main idea is to use one integration method with different step-sizes and then combine the resulting solutions in order to obtain a new estimate with better numerical accuracy. Here, the underlying integration method in many cases is a linear multistep method but may also be e. g. a RUNGE-KUTTA method. Two different approaches are to be distinguished between: one may or may not use extrapolated values for calculations in further time steps. Often, these methods are then called 'active' or 'passive', respectively (with a different meaning of passivity than being used in this paper). We will only consider the second variant here and will merely speak of extrapolation from now on.

Assume that m different numerical solutions of the differential eq. (1a) have been obtained by (independently) applying one and the same numerical integration method of order q with different step-sizes T/n_1, T/n_2, ... T/n_m and denote by v_μ the corresponding approximate solution for $\tilde{v}$ at some specific time instance. With these values, one may construct a suitable interpolation polynomial $u(\tau)$ possessing the m specific values $u(T/n_\mu) = v_\mu$. Then, the extrapolated value $u(0)$ can be used as an estimate $\hat{v}$ with an order increased by $m-1$ at least.

Several means of obtaining this estimate are known from literature. One possible way of proceeding is analogue to that having been, in context with the frequency-based approach, presented in section 3.2. In generalization of eq. (6), one may write

$$v_\mu = \tilde{v} + \sum_{\lambda=1}^{m-1} \hat{\varepsilon}_\lambda [T/n_\lambda]^{q+\lambda-1} + \mathcal{O}\left(T^{q+m-1}\right) \tag{33}$$

with some error constants $\hat{\varepsilon}_\lambda$ [12]. Now, the goal is to linearly combine the values v_μ in order to find an estimate $\hat{v}$ of order $q+m-1$:

$$\hat{v} = \sum_{\mu=1}^{m} g_\mu v_\mu = \tilde{v} + \mathcal{O}\left(T^{q+m-1}\right). \tag{34}$$

A comparison with eqs. (23) and (25) shows that one merely has to replace $Y_{\mu,\nu}$ by $v_{\mu,\nu}$ with $v_{\mu,1} := v_\mu$ and n_μ by $1/n_\mu$ throughout in order to obtain the sought estimate $\hat{v} = v_{mm}$. In particular, eqs. (26) and (27) now read

$$v_{\mu,\nu+1} = v_{\mu,\nu} + \frac{v_{\mu,\nu} - v_{\mu-1,\nu}}{\left[n_\mu \,/\, n_{\mu-\nu}\right] - 1} \tag{35}$$

for first order methods and

$$v_{\mu,\nu+1} = v_{\mu,\nu} + \frac{v_{\mu,\nu} - v_{\mu-1,\nu}}{\left[n_\mu \,/\, n_{\mu-\nu}\right]^2 - 1} \tag{36}$$

for methods like the trapezoidal rule with (35) being known as the AITKEN-NEVILLE algorithm for calculating the value of an interpolation polynomial at the point 0 [21, 22].

Different types of sequences n_1, n_2, ... can be found in literature. The most common ones are the harmonic and the ROMBERG sequence,

$$n_\mu = \mu \quad \text{and} \quad n_\mu = 2^{\mu-1}, \tag{37}$$

as well as the BULIRSCH sequence

$$n_\mu = \begin{cases} 1 & \text{for } \mu = 1 \\ 2^{\mu/2} & \text{for } \mu \text{ even} \\ 3 \cdot 2^{(\mu-3)/2} & \text{for } \mu \neq 1 \text{ odd} \end{cases}. \tag{38}$$

5. Application to wave digital simulation

Among the large variety of concepts for simulating KIRCHHOFF networks, the wave digital concept offers some specific features making it an interesting alternative to more classical approaches and we will make use of it in the following section. Clearly, a complete coverage of this concept cannot be part of this paper and the reader is referred to the existing literature (e. g. [1]). At this point, it is only to be mentioned that the wave digital concept is, in particular, component-oriented and thus allows for a systematic generation of algorithmic models. Moreover, passivity of these models is always implied by the passivity of the specific components which can be guaranteed by simple means even under finite wordlength conditions.

However, one can only take advantage of the latter property as long as the numerical integration in use does not produce any form of 'activity' itself. This means that the frequency-based approach of section 3 cannot, at least in combination with linear multistep methods, be used with increased accuracy for problems where passivity is a crucial feature.

The situation turns out to be different when it comes to applying extrapolation methods in the form of section 4. Note that, with regard to the nature of the underlying integration method, the extrapolation requires no other assumption than the existance of the expression (33). In particular, no detailed knowledge is necessary about the error constants $\hat{\varepsilon}_\lambda$ whose determination is rather envolved [14]. One may hence as well start from a wave digital simulation with one of the methods represented in Fig. 1. Now,

no kind of feedback into the algorithmic models running at different step-sizes occurs (the extrapolation is only applied to the simulation results being obtained independently). Thus, stability is a consequence of the stability of the individual models which is now guaranteed by applying a passive basis method.

6. Two examples

In this section, two examples based on wave digital simulation principles will be presented. In both cases, the solution of the underlying differential equation is known and can be written in closed form. With this, the claimed gain in accuracy can be verified by calculating the overall simulation error.

6.1 An ordinary differential equation

Consider the KIRCHHOFF network shown in Fig. 4 containing a resistive voltage source, a resistance, a capacitance, and an ideal diode. The way from this circuit to the corresponding wave digital structure consists of the following basic steps [1]:

- Decompose the circuit into port-wise interconnected components.
- Assign a (suitably chosen) positive *port resistance* R_ν to each port ν.
- At each port with port voltage v_ν and port currrent i_ν, introduce *(voltage) wave quanitities*

$$a_\nu := v_\nu + R_\nu i_\nu \quad \text{and} \quad b_\nu := v_\nu - R_\nu i_\nu \,.$$

- Use these port quantities instead of voltages and currents to describe dynamic-free elements like resistive source, resistance, and diode. For example, the ideal diode being described by

$$v_\mathrm{D} = 0 \;\text{ for }\; i_\mathrm{D} \geq 0 \quad \text{and} \quad i_\mathrm{D} = 0 \;\text{ for }\; v_\mathrm{D} \leq 0$$

possesses the wave digital description

$$b_\mathrm{D} = -|a_\mathrm{D}|$$

independently of the chosen port resistance [3].
- Additionally apply a passive integration method to the differential equations describing reactive elements. For a capacitance, an application of the trapezoidal rule with port resistance $T/[2C]$ leads to

$$b(t) = a(t - T)$$

and hence to a delay element as a wave digital realization.
- Use so-called *adaptors* like e. g. the parallel or series adaptor for the wave digital realization of the KIRCHHOFF laws. According to this procedure, the wave digital structure shown in Fig. 5 is obtained.

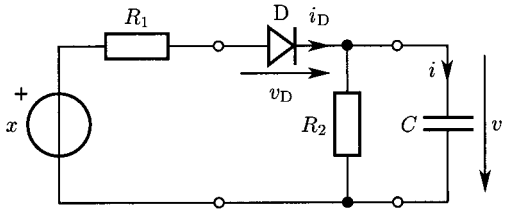

Fig. 4. Circuit example.

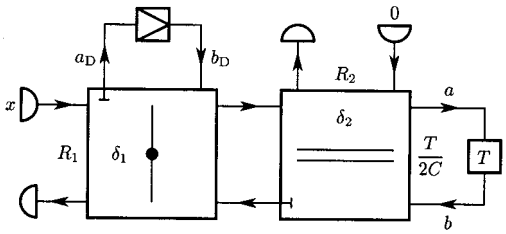

Fig. 5. Wave digital structure according to the circuit depicted in Fig. 4.

Now, the circuit of Fig. 4 with initial value $v(0) = 0.8\,\mathrm{V}$ has been simulated over the interval $0\,\mathrm{s} \leq t \leq 2\,\mathrm{s}$ where the input x has been periodic with period 1 s and

$$x(t) = \begin{cases} 1\,\mathrm{V} \text{ for } 0\,\mathrm{s} < t < 0.5\,\mathrm{s} \\ 0\,\mathrm{V} \text{ for } 0.5\,\mathrm{s} < t < 1\,\mathrm{s} \end{cases}$$

as shown in Fig. 6 while the network parameters were $R = 1\ \Omega$ and $C = 1$ F. The exact output $\tilde{v}$ of the circuit is depicted in Fig. 7.

The first diagram in Fig. 8 shows the overall simulation error obtained via the structure in Fig. 5 with an initial step-size $T = 1/F = 0.1$ s. Simulations with step-sizes $T/2$ and $T/4$ (using the ROMBERG sequence) yielded the further results shown below. We see that the (global) simulation error is reduced by a factor four each time the step-size is bisected. This behavior is consistent with the order two of the underlying trapezoidal rule. The remaining diagrams show the drastically reduced simulation error obtained by extrapolation where, for the one-step extrapolated method as shown in the second column, a bisection of the step-size reduces the global error by a factor of about 16 which again is consistent with

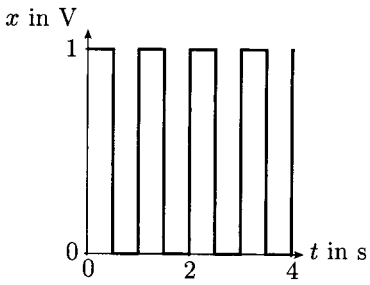

Fig. 6. Input signal for the circuit shown in Fig. 4.

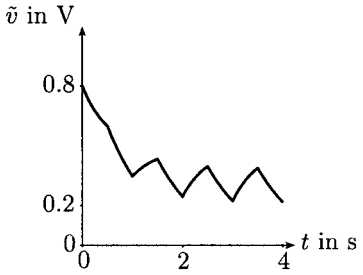

Fig. 7. Exact solution for the circuit shown in Fig. 4 with input according to Fig. 6.

the order four being achieved this way. Further simulations confirm that a simulation with step-size $T/8$ (with an overall implementation effort being roughly comparable to the three simulations plus extrapolation) produces a maximum error of approximately $6.25 \cdot 10^{-6}$ V, i. e. one fourth of that obtained with $T/4$. In comparison with this, extrapolation has improved the numerical accuracy by a factor of about 35000. On the other hand, an error comparable to the one obtained with the extrapolated method requires a single run with an approximate step-size $T/1400$ (with accordingly higher memory requirements). In our case, extrapolation lead to a total reduction in simulation time by a factor 40. Here, we have only compared numerical results on the coarsest grid.

6.2 A partial differential equation

As the following example will show, the application of extrapolation methods to wave digital simulation becomes even more attractive in case of partial differential equations. Consider the telegraphists equations

$$ri + l\frac{\partial i}{\partial t} + \frac{\partial v}{\partial z} = u \tag{39a}$$

$$gv + c\frac{\partial v}{\partial t} + \frac{\partial i}{\partial z} = j \tag{39b}$$

with constant parameters $l > 0$, $c > 0$, $r \geq 0$, and $g \geq 0$ as well as excitations u and j. The sought values v and i depend on time t and one spatial coordinate, here denoted as z.

Different (multidimensional) KIRCHHOFF networks realizing the telegraphists equations are known, e. g. [4]. We will consider the one depicted in Fig. 9. Here, voltages and currents at the capacitances c_+ and c_- are related by

$$i_+ = \frac{c}{2}\left(\frac{\partial v_+}{\partial t} + v_0\,\frac{\partial v_+}{\partial z}\right) \tag{40a}$$

$$i_- = \frac{c}{2}\left(\frac{\partial v_-}{\partial t} - v_0\,\frac{\partial v_-}{\partial z}\right) \tag{40b}$$

with some positive constant v_0 (with the dimension of a velocity, not to be confused with the initial voltage value in eq. (1a)). An analysis of the circuit leads to the eqs. (39) if one sets

$$v_0 = \frac{1}{\sqrt{lc}} \quad \text{and} \quad r_0 = \sqrt{\frac{l}{c}} \tag{41}$$

plus

$$v_+ = v + r_0 i \quad \text{and} \quad v_- = v - r_0 i\,. \tag{42}$$

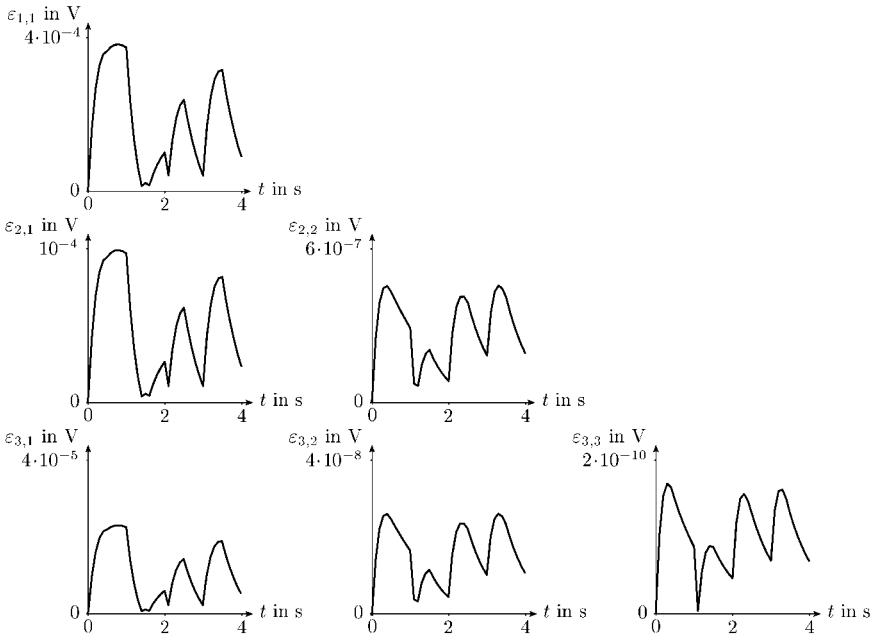

Fig. 8. Deviation $\varepsilon_{\mu,\nu} := |v_{\mu,\nu} - \tilde{v}|$ for the extrapolated trapezoidal rule based on ROMBERG's sequence.

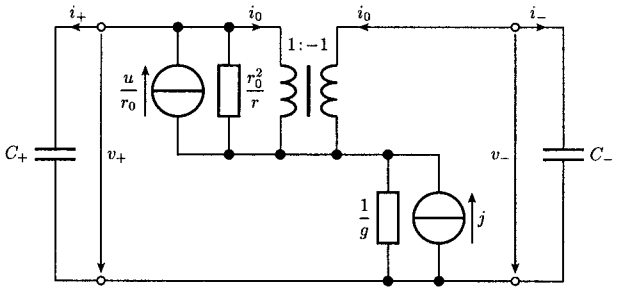

Fig. 9. Circuit representation of the telegraphists equations (39).

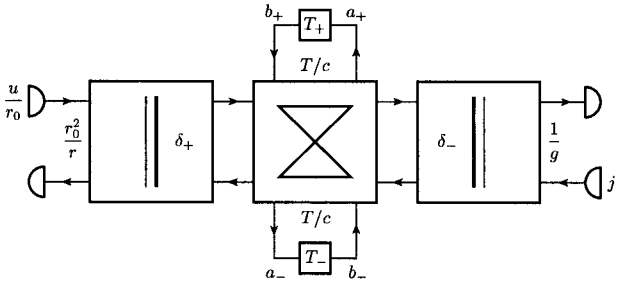

Fig. 10. Wave digital structure corresponding to Figure 9.

The way from the KIRCHHOFF network to the corresponding wave digital structure shown in Fig. 10 is analogue to the one outlined before using, in particular, a so-called lattice adaptor for the JAUMANN transformer four-port [1]. However, one has to apply the multidimensional trapezoidal rule to equations (40) such that they are replaced by

$$b_+(t,z) = a_+(t-T, z-Z) \tag{43a}$$
$$b_-(t,z) = a_-(t-T, z+Z) \tag{43b}$$

for port resistance T/c where spatial and time delay are related by

$$Z = v_0 T\,. \tag{44}$$

But, a spatial delay can be interpreted as shift of location in memory. Performing this 'uncoiling', one gets the conventional wave digital structure displayed in Fig. 11.

Now, a telegraph line with zero initial state has been simulated for the time interval $0\,\mathrm{s} \le t \le 2\,\mathrm{s}$ over the spatial

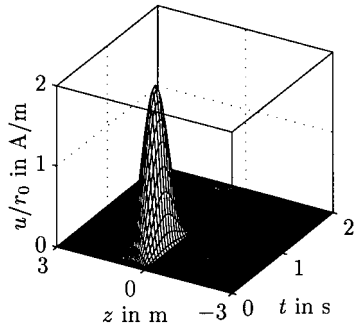

Fig. 12. Product of two $\sin^2$-pulses used for the exciation of the telegraphists equations.

range $-3\,\mathrm{m} \le z \le 3\,\mathrm{m}$. The input signals were

$$u(t,z) = 2\,\mathrm{V/m} \cdot \mathrm{sint}(t/\mathrm{s})\mathrm{sint}(z/\mathrm{m})$$

and $j(t,z) = 0\,\mathrm{A/m}$ with

$$\mathrm{sint}(\xi) := \sin^2(\pi\xi)\mathrm{rect}\,(2\xi - 1)\ ,$$

i. e. the excitation in (39a) is the product of two $\sin^2$-pulses with duration 1 s and spread 1 m as shown in Fig. 12. The line parameters were chosen to be $l = 1\,\mathrm{H/m}$, $c = 1\,\mathrm{F/m}$, $r = 1\,\Omega/\mathrm{m}$, and $g = 1\,\mathrm{S/m}$. Note that the parameters fulfill the relation $lg = rc$. The line is thus *distortion-free* and the solution of eqs. (39) can be calculated in closed form, the result depicted in Fig. 13. Moreover, the line possesses a real wave resistance

$$Z_\mathrm{w} = \sqrt{\frac{r + \mathrm{j}\omega l}{g + \mathrm{j}\omega c}} = \sqrt{\frac{l}{c}} = r_0$$

and can be terminated without any reflection by attaching resistances r_0 on both ends $z = z_\mathrm{l}$ and $z = z_\mathrm{r}$. For $z = z_\mathrm{l}$, there then holds $v = -r_0 i$, eq. (42) implies $v_+ = 0$ and with $v_+ = [i_+ - i_0]r_0^2/r + [i_+ + i_0]/g$ for zero source-values also $i_+ = 0$. With this, the (necessarily) finite spatial simulation range does not produce any artificial reflection in the numerical solution if one merely sets $a_+(t, z_\mathrm{l}) = 0$ and likewise $a_-(t, z_\mathrm{r}) = 0$.

Figure 14 shows simulation results with different step-sizes and extrapolation. The initial step-size was 0.05 s and the ROMBERG sequence has been used again. For

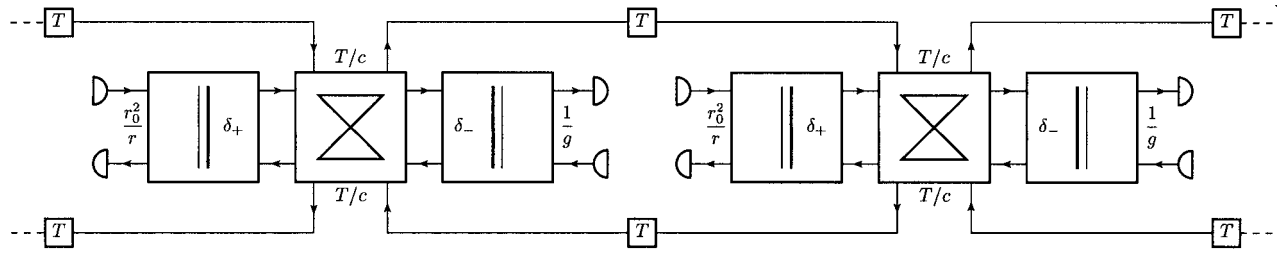

Fig. 11. Uncoiled wave digital structure corresponding to Fig. 10.

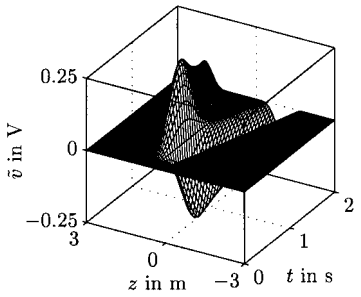

Fig. 13. Exact solution of the telegraphists equations for the excitation given in Fig. 12.

a comparison with a single simulation with reduced step-size, one has to note that, due to the coupling (44) of step-size and spatial resolution, the implementation effort increases quadratically with $1/T$. We may thus claim that a simulation run with step-size $T/5$ requires more effort than the aforementioned three runs plus extrapolation (with $1+2^2+4^2<5^2$). Yet, the error is approximately 35000 times larger than that of the extrapolation method. This is about the same accuracy gain as has been found for the ordinary differential equation in the previous example. However, a simulation at the step-size required for an error comparable to that of the extrapolated method (an estimated $T/900$) could not be performed within the Matlab® environment used by the authors due to the quadratic increase in implementation effort.

7. Conclusions

This paper has dealt with concepts for increasing the numerical accuracy attainable with wave digital simulation techniques. It has been shown that extrapolation techniques known from numerical mathematics provide an effective means of achieving this goal, especially for the trapezoidal rule being classically used within the wave digital concept. This is done without sacrifycing the key stability properties of a wave digital simulation (induced by passivity) because the methods considered here do

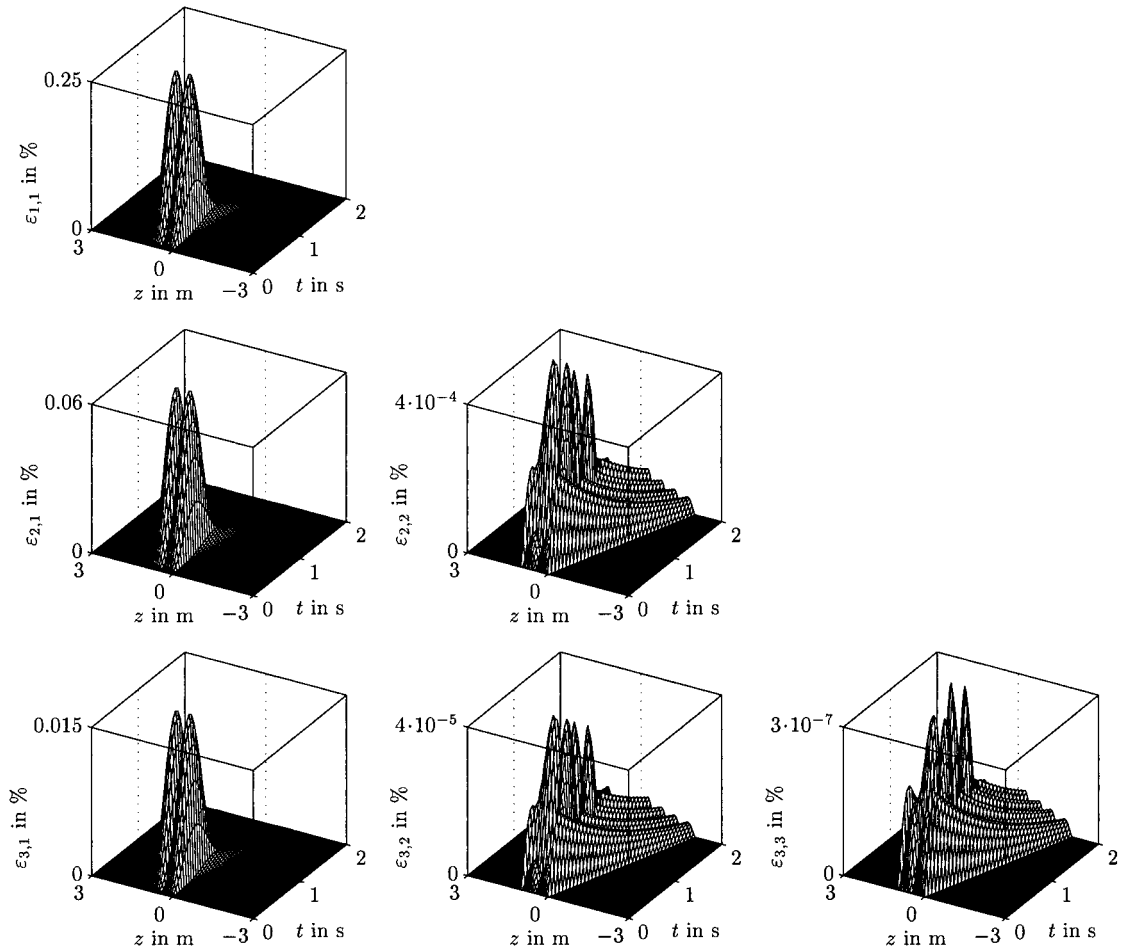

Fig. 14. Relative error $|v_{\mu,\nu}-\tilde{v}|/\max_{t,z}|\tilde{v}|$ in % for the extrapolated trapezoidal rule based on ROMBERG's sequence.

not reuse any of the extrapolated values. Although only treated in context with wave digital models based on linear multistep methods, extrapolation clearly can also be combined with e. g. higher order passive RUNGE-KUTTA methods.

Regardless of the underlying numerical integration method, extrapolation can be used to increase the attainable consistency order beyond the limits imposed on the pure method by the demand for passivity (the maximum order is two for passive linear multistep methods and can only be increased to four for suitable passive RUNGE-KUTTA methods). Such an increase is particularly desirable in case of partial differential equations where memory requirements and implementation effort may otherwise prevent a successful wave digital simulation when very numerical high accuracy is demanded. This applies all the more because, even for low order partial differential equations, a wave digital model based on a RUNGE-KUTTA method in general is hard if not impossible to find.

In contrast to the techniques without reuse of extrapolated values, the frequency-based approach itself does not provide an increase of accuracy for passive methods. However, it offers a first insight into simulation techniques based on methods with different step-sizes and may prove helpful in investigating techniques reusing extrapolated values. From the viewpoint of digital signal processing, these techniques relate to multi-rate systems and a corresponding analysis is part of ongoing research.

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