Divide-and-conquer: Approaching the capacity of the two-pair bidirectional Gaussian relay network

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Abstract

In this paper we study the capacity region of the multi-pair bidirectional (or two-way) wireless relay network, in which a relay node facilitates the communication between multiple pairs of users. This network is a generalization of the well known bidirectional relay channel, where we have only one pair of users. We first examine this problem in the context of the deterministic channel interaction model, which eliminates the channel noise and allows us to focus on the interaction between signals. We characterize the capacity region of this network when the relay is operating at either full-duplex mode or half-duplex mode (with non adaptive listen-transmit scheduling). In both cases we show that the cut-set upper bound is tight and, quite interestingly, the capacity region is achieved by a divide-and-conquer strategy using signal level alignment, in which the relay applies a simple equation-forwarding scheme. We then use the insights gained from the deterministic network for the Gaussian two-pair full-duplex directional relay network. The strategy in the deterministic channel translates to a specific superposition of lattice codes and random Gaussian codes at the source nodes with alignment in signal scale for the Gaussian network. The relay attempts to decode the Gaussian codewords and the superposition of the aligned lattice codewords of each pair. Then it forwards this information to all users. We analyze the achievable rate of this scheme when there are only two pairs and show that for all channel gains

Parts of this paper have been presented at ITW'09 [1] and ISIT'09 [2], respectively.

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it achieves to within 2 bits/sec/Hz per user of the cut-set upper bound on the capacity region of the two-pair bidirectional Gaussian relay network.

I. Introduction

Cooperative communication and relaying is one of the important research topics in wireless network information theory. The basic model to study this problem is the 3-node relay channel which was first introduced in 1971 by van der Meulen [3] and the most general strategies for this network were developed by Cover and El Gamal [4].

While the main focus so far is on the one-way-relay channel, bidirectional communication has also attracted attention. Bidirectional or two-way communication between two nodes was first studied by Shannon himself in [5]. Nowadays the two-way communication where an additional node acting as a relay is supporting the exchange of information between the two nodes (or one pair) is gaining increased attention. Some relaying strategies for this one-pair two-way relay channel, such as decode-and-forward, compress-and-forward and amplify-and-forward, have been analyzed in [6]. Network coding type techniques have been proposed by [7], [8], [9], [10] (and others) in order to improve the transmission rate.

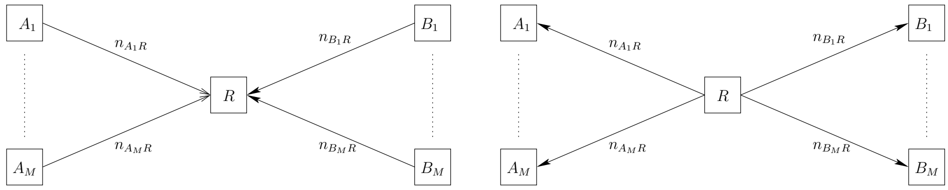
The two way relay channel problem can be generalized to a multi-pair (or multiuser) setting in which the relay facilitates the communication between multiple pairs of users. In [11] authors analyzed the case that the relay orthogonalizes different two-way transmissions by a distributed zero forcing algorithm and then multiple pairs communicate with each other via several orthogonalize-and-forward relay terminals. In [12], [13] authors investigated this problem for interference limited systems in which each pair of users share a common spreading signature to distinguish themselves from the other pairs, and proposed a jointly demodulate-and-XOR forward strategy. However, so far no attempt has been done to characterize the capacity region of this network, and the optimal relaying strategy is unknown.

In this paper we study the information theoretic capacity of the multi-pair bidirectional wireless relay network. We first examine this problem in the context of the deterministic channel interaction model. The deterministic model, studied by Avestimehr, Diggavi, and Tse [14], simplifies the wireless network interaction model by eliminating the noise and allows us to focus on the interaction between signals. This approach was successfully applied to the relay network in [14], and resulted in insight in terms of transmission techniques which further led to an approximate characterization of the noisy wireless relay network problem [15]. This approach has also been recently applied to the bidirectional relay channel problem [16], [17], which again resulted in finding near optimal relaying strategies as well as approximating the capacity region of the noisy (Gaussian) bidirectional relay channel.

Inspired by these results, we apply the deterministic model to the multi-pair bidirectional relay network and analyze its capacity when the relay is operating at either full-duplex mode or half-duplex mode (with non adaptive listen-transmit scheduling). In both cases we exactly characterize the capacity region and show that the cut-set upper bound is tight. We also make use of the recent idea of interference alignment introduced by [18], [19]. In those works, interference alignment is performed in signal space; in this paper, it is achieved in signal level (and later on in the Gaussian setup equivalently in signal scale). In more details, we show that the capacity region is achieved by dividing the signal level space elegantly between the multiple pairs, i.e. different pairs are orthogonalized on the signal level space. Each pair is then operating on the portion signal level space assigned to it. The strategy is therefore referred to as divide-and-conquer-strategy with signal alignment at signal level. The relay uses a similar *equation-forwarding* scheme as in [16], in which the relay re-orders the received superposed signals on the different levels and forwards them.

Later on, we use these insights to find a near optimal transmission technique for the Gaussian case. More specifically, we propose a superposition of lattice codes and random Gaussian codes at the source nodes. The relay attempts to decode the Gaussian codewords and the superposition of the aligned lattice codewords of each pair. The relay then forwards this information to the intended destinations. We analyze the achievable rate region of this scheme and show that for all channel gains it achieves to within 2 bits/sec/Hz per user of the cut-set upper bound on the capacity region of the two-pair bidirectional relay network.

The paper is organized as follows. In Section II we investigate the full-duplex and half-duplex multi-pair bidirectional deterministic relay network and characterize the exact capacity region of this network. In Section III, we discuss the insights gained from the deterministic model and how these insights can be used in the Gaussian setup in the subsequent Section IV. In the Gaussian two-pair two way relay network, we present upper bounds, derive our achievability strategy and characterize the constant gap between the upper bounds and our proposed scheme. We finally conclude the paper in Section V.



(a) Uplink (b) Downlink

Fig. 1. The system model for M pair bidirectional deterministic relay network.

II. Multi-pair bidirectional deterministic relay network

In the following subsections, we state the precise definition of the problem and present the main result for the deterministic case.

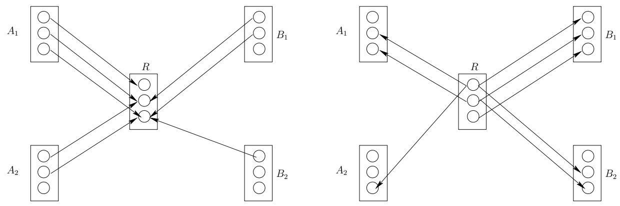
A. System model

The system model for the M-pair bidirectional relay network is shown in Figure 1. In this system M pairs $(A_1, B_1), \dots, (A_M, B_M)$ aim to use the relay to communicate with each other (*i.e.* A_1 and B_1 want to communicate with each other, and so on). The relay can operate on either full-duplex or half-duplex mode. In the full-duplex mode it is able to listen and transmit at the same time, while in the half-duplex mode it can only listen or transmit at a particular time. In the half-duplex scenario, we only consider the case that the listen-transmit scheduling is non-adaptive and the relay listens a fixed t fraction of the time and transmits the rest. Although t can not change adaptively as a function of the channel gains, one can optimize over t beforehand.

We use the deterministic channel model to model the interaction between the transmitted signals. The deterministic channel model was introduced in [14]. Here is a formal definition of this channel model.

Definition II.1: (Definition of the deterministic model) Consider a wireless network as a set of nodes V, where $|V| = N$. Communication from node i to node j has a non-negative integer gain[1] $n_{(i,j)}$ associated with it. This number models the channel gain in a corresponding Gaussian setting. At each time t, node i transmits a vector $\mathbf{x}_i[t] \in \mathbb{F}_2^q$ and receives a vector $\mathbf{y}_i[t] \in \mathbb{F}_2^q$ where $q = \max_{i,j}(n_{(i,j)})$. The received signal at each node is a deterministic function of the

[1]Some channels may have zero gain.



(a) Uplink (b) Downlink

Fig. 2. The pictorial representation of a two-pair bidirectional deterministic relay network with channel gains $n_{A_1R} = 3$, $n_{B_1R} = 2$, $n_{A_2R} = 2$, $n_{B_2R} = 1$, $n_{RA_1} = 2$, $n_{RB_1} = 3$, $n_{RA_2} = 1$ and $n_{RB_2} = 2$.

transmitted signals at the other nodes, with the following input-output relation: if the nodes in the network transmit $\mathbf{x}_1[t], \mathbf{x}_2[t], \ldots \mathbf{x}_N[t]$ then the received signal at node j, $1 \leq j \leq N$ is:

$$\mathbf{y}_j[t] = \sum_{k=1}^{N} \mathbf{S}^{q-n_{k,j}} \mathbf{x}_k[t] \tag{1}$$

for all $1 \leq k \leq N$, where $\mathbf{S}$ is the $q \times q$ shift matrix and the summation and multiplication is in $\mathbb{F}_2$.

Now that we have defined the deterministic channel model we can apply it to the multi-pair bidirectional relay network. A pictorial representation of an example of such network with two pairs is shown in Figure 2. In this figure each little circle represents a signal level and what is sent on it is a bit. The transmit and received signal levels are sorted from MSB to LSB from top to bottom. The channel gain between two nodes i and j indicates how many of the first MSB transmitted signal levels of node i are received at destination node j. As described in the channel model (1), at each received signal level, the receiver gets only the modulo two summation of the incoming bits.

B. Capacity region of the deterministic relay network

In this section we study the capacity region of the multi-pair bidirectional deterministic relay network. First we state the main result for the deterministic network and then in the rest of this section prove it.

Theorem 1: The capacity region of the multi-pair bidirectional deterministic relay network (both full-duplex and half-duplex), described in Section II-A, is equal to the cut-set upper bound region, and it is achieved by a divide-and conquer-strategy using signal level alignment with a simple equation forwarding relaying scheme.

For example in the case that we have only two pairs and the relay is operating on the full-duplex mode, the cut-set upper bound on the capacity region is given by

$$R_{A_1} \leq \min\left(n_{A_1R}, n_{RB_1}\right) \tag{2}$$

$$R_{B_1} \leq \min\left(n_{B_1R}, n_{RA_1}\right) \tag{3}$$

$$R_{A_2} \leq \min\left(n_{A_2R}, n_{RB_2}\right) \tag{4}$$

$$R_{B_2} \leq \min\left(n_{B_2R}, n_{RA_2}\right) \tag{5}$$

$$R_{A_1} + R_{A_2} \leq \min\left(\max\left(n_{A_1R}, n_{A_2R}\right), \max\left(n_{RB_1}, n_{RB_2}\right)\right) \tag{6}$$

$$R_{B_1} + R_{B_2} \leq \min\left(\max\left(n_{B_1R}, n_{B_2R}\right), \max\left(n_{RA_1}, n_{RA_2}\right)\right) \tag{7}$$

$$R_{A_1} + R_{B_2} \leq \min\left(\max\left(n_{A_1R}, n_{B_2R}\right), \max\left(n_{RB_1}, n_{RA_2}\right)\right) \tag{8}$$

$$R_{B_1} + R_{A_2} \leq \min\left(\max\left(n_{B_1R}, n_{A_2R}\right), \max\left(n_{RA_1}, n_{RB_2}\right)\right). \tag{9}$$

Now consider the network shown in Figure 2. It is easy to check that the rate tuple

$$\left(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2}\right) = (2, 1, 1, 1) \tag{10}$$

is inside its cut-set region. In Figure 3 we illustrate a simple scheme that achieves this rate point. With this strategy, the nodes in the uplink transmit

$$x_{A_1} = \left[a_{1,1}, a_{1,2}, 0\right]^t, \quad x_{B_1} = \left[b_{1,1}, 0, 0\right]^t$$

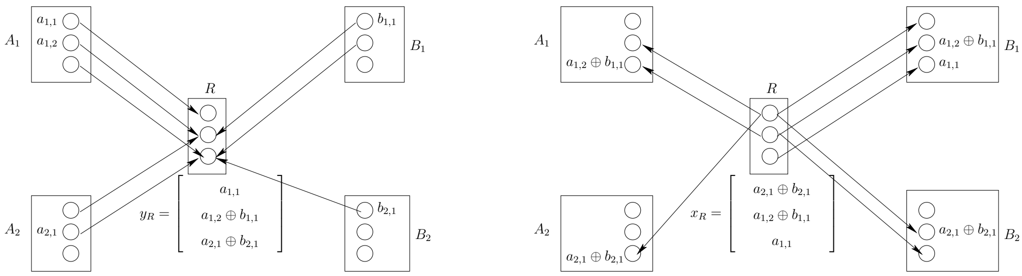
$$x_{A_2} = \left[0, a_{2,1}, 0\right]^t, \quad x_{B_1} = \left[b_{2,1}, 0, 0\right]^t$$

and the relay receives

$$y_R = [a_{1,1},\ a_{1,2} \oplus b_{1,1},\ a_{2,1} \oplus b_{2,1}]^t.$$

Then the relay will re-order the received superposition of the signals of both users of each pair (referred to as equations in the following, as e.g. $a_{1,2} \oplus b_{1,1}$) and transmit

$$x_R = [a_{2,1} \oplus b_{2,1},\ a_{1,2} \oplus b_{1,1},\ a_{1,1}]^t.$$



(a) Uplink (b) Downlink

Fig. 3. The scheme that achieves rate point $(2, 1, 1, 1)$.

Then node A_1 receives the superposed signals (i.e equation) $a_{1,2} \oplus b_{1,1}$ and since it knows $a_{1,2}$ can decode $b_{1,1}$, similarly node B_1 can decode $a_{1,1}$ and $a_{1,2}$, node A_2 can decode $b_{2,1}$ and finally node B_2 can decode $a_{2,1}$. Therefore we achieve the rate point $(2, 1, 1, 1)$.

There are some interesting points about this particular achievability strategy,

- There is no coding over time,
- There is no interference between different pairs on the same received signal level at the relay,
- Signal from the same pair arrive at the same levels at the relay, i.e. signal level alignment is performed.
- The relay just re-orders the received equations and forwards them.

We call a strategy with these properties an *divide-and-conquer* strategy with an *equation forwarding* relaying scheme. A natural question is whether we can always achieve any rate point in the cut-set bound region with a divide-and-conquer strategy?

Quite interestingly, in the next lemma we show that indeed this is always possible.

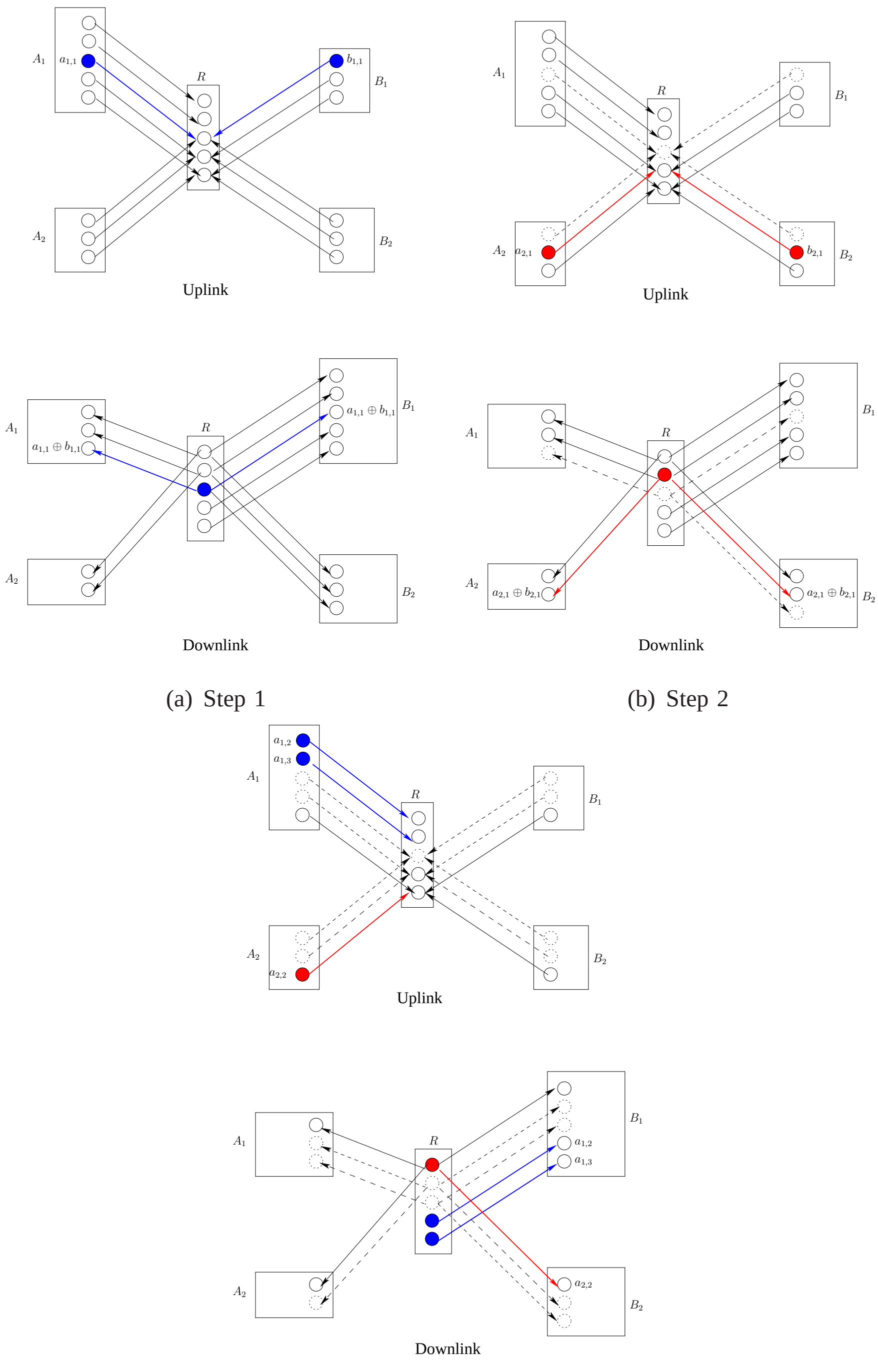
Lemma 1: Any integral[2] rate tuple inside the cut-set upper bound on the capacity region of the full-duplex M-pair bidirectional deterministic relay network is achieved by a divide-and-conquer strategy with a simple equation forwarding relaying scheme.

The proof is given in Appendix A.

[2]with integer components.

Now to prove the main Theorem 1 for the full-duplex scenario, we just need to show that all corner points of the cut-set bound region are achieved by a divide-and-conquer strategy. Note that since all coefficients of the hyperplanes of the cut-set bound region are integers, then all corner points of the region must be fractional. If a corner point $\overrightarrow{R}$ is integral then by Lemma 1 we know that it is achieved by a divide-and-conquer strategy. If it is not integral then choose a large enough integer Q such that $Q\overrightarrow{R}$ is integral. Now note that Q instances of a deterministic network over time is the same as the original network except all channel gains are multiplied by Q. Now since $Q\overrightarrow{R}$ is integral and is obviously inside the cut-set upper bound of the big network (where all channel gains are multiplied by Q), then by Lemma 1 it is achievable by a divide-and-conquer strategy. This strategy can then be simply translated to a divide-and-conquer strategy on the original network over Q time-steps. Therefore the corner point $\frac{QR}{Q} = R$ is achievable.

Similarly we can also prove Theorem 1 for the half-duplex scenario. In this case relay listens t fraction of the time and transmits the rest. Without loss of generality assume t is a fractional number (otherwise consider the sequence of fractional numbers approaching it). Then choose a large enough integer Q such that Qt is integer. Then consider Q instances of the network over time, such that for Qt instances the relay is listening and in the other $Q(1-t)$ instances it is transmitting. After concatenating these instances together, the resulting network can be thought of as a full-duplex multi-pair network where the uplink channel gains are multiplied by Qt and the downlink channel gains are multiplied by $(1-t)Q$. It is easy to verify that the cut-set bound region of this network is just the cut-set bound region of the original half-duplex network expanded by Q. Now by Lemma 1 and the previous argument, we know that the capacity region of this full-duplex multi-pair bidirectional network is equal to its cut-set upper bound region and is achieved by a divide-and-conquer strategy with a simple equation forwarding scheme. Now note that any divide-and-conquer strategy in this full-duplex network can be translated to a divide-and-conquer strategy in Q instances of the original half-duplex network; Qt instances the relay is in the listen mode to get the equations and $(1-t)Q$ instances in the transmit mode to send the equations. Therefore the cut-set upper bound region is achievable and the proof is complete.



(a) Step 1 (b) Step 2

(c) Step 3

Fig. 4. Illustration of the inductive algorithm introduced in Lemma 1

C. Remark

Although the scheme we provided is an inductive way of level assignment and seems quite unstructured (in the sense that it assigns signal levels on a greedy basis), one can actually say more about these assignments using certain observations. First of all, note that in this divide-and-conquer strategy we have in general $2M$ types of signals that the relay might decode. Namely, M types of signals that are made up of one bit from one user of a session[3], and M types of signals that are the superposition of bits from both users of the same pair. Each signal is received at the relay at some signal level, and is transmitted to one or both of the end users at potentially another signal level in down-link. Please refer to the example network of Figures 4 and 5, and observe that quite interestingly, in the final configuration of signal type-level assignments, all signals of the same type are concatenated together both in UL and DL. In other words they appear at concatenated signal levels. In general, one can serve all signals of the same type at once by choosing a pair with nonzero rates and serve them (one bit per user per signal level) until one of the rates is zero. For example, assuming $R_{A_1} \leq R_{B_1}$ and $R_{A_2} \leq R_{B_2}$, instead of reducing (R_{A_1},R_{B_1},R_{A_2},R_{B_2}) to ($R_{A_1} - 1$,$R_{B_1} - 1$,R_{A_2},R_{B_2}) one can reduce it to (0,$R_{B_1} - R_{A_1}$,R_{A_2},R_{B_2}) all at once and a find a chunk of signal levels to afford them. Then the same thing can be done for the other pair. Now by rearranging the signal levels, it is easy to show that in the final configuration, all signals of the same type are in concatenation. The conclusion of the above argument is that for any network configuration, one can always find $2M$ groups of disjoint signal levels (both in UL and DL) with the following properties:

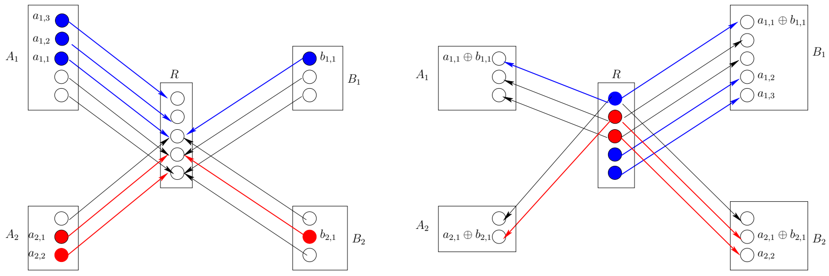
$$\vdots$$

- Group $2i - 1$: $\min\{R_{A_i}, R_{B_i}\}$ levels connected to both A_i and B_i.
- Group $2i$: $\max\{R_{A_i}, R_{B_i}\} - \min\{R_{A_i}, R_{B_i}\}$ levels connected to the one of A_i and B_i with higher transmission rate.

$$\vdots$$

Furthermore, levels of each group are all in concatenation. Intuitively, this observation suggests that in the noisy channel case, each user need to only break the transmit signal to at most M

[3] A session means the communication of one pair



(a) Uplink (b) Downlink

Fig. 5. Illustration of the resulting divide-and-conquer strategy of the inductive algorithm.

different signal levels, independent of the channel gains. The importance of this issue comes up when we talk about the Gaussian channel model, in which we have to take into account the effect of noise. In that case, an inductive strategy, similar to the deterministic network case, would fail because every signal level split-up requires one bit draw back in the achievable rate region to cancel out the effects of noise and interference appropriately. If one has to do the slip-up for every single equation transmitted, this will make the theoretical achievable rate region useless. However, using the aforementioned observation one would only need M split-ups in the signal level, and this means that the cut set bound is achievable within M bits. If M is small, e.g $M = 2$, this would be a tight lower bound for the achievable rate region.

In the following, we discuss more insights gained from the examination of the deterministic multi-pair two-way relay network that can be interpreted for the two-pair Gaussian relay network.

III. Transition from the deterministic model to Gaussian model

The result of the deterministic network basically suggests that it is optimal to divide the signal-level space into subspaces and allocate these orthogonal subspaces to the different sessions. Furthermore, it suggests to split the message of the stronger user of each pair (the user with stronger uplink channel) into two parts:

1) the first part has the same rate as the rate as the message from the weak user and it is transmitted such that at the relay it is received with the same power (i.e. aligned in signal scale) as that of the signal from the weak user,
2) the second part has the remaining rate and is transmitted at some higher signal levels.

Hence, for $M = 2$, the relay receives four chunks of bits at different signal levels. Namely, the bits that are created from the superposition of the signals of both users of each pair and, bits from the signals of the strong transmitter of each pair. The relay then forwards these signals at non-overlapping signal levels to the end users so that the superposed signals (i.e. equations) are received by both users, whereas the other bits (from the strong transmitters) are received by the corresponding end users only. This way each user can easily decode its message having the received equations, received bits and its own transmitted message.

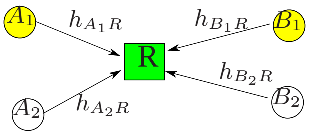
To apply a similar strategy to Gaussian networks, one will face three immediate challenges. The first one is the effect of the additive noise which is inevitably present in the Gaussian channels. The second issue is the power leakage from the signals of lower levels (e.g. superposition of chunks of signals) to those transmitted at higher levels; should one try to break the messages into superpositions of low power and high power signal as in the deterministic case. The third complication is in decoding the equations (i.e. superposition of signals) which should take place at the relay.

We propose the following solutions to overcome these difficulties. The noise issue can be simply resolved by using an appropriate block symbol coding scheme. The leakage problem is inevitable, since in the wireless Gaussian channel the interference will always exist. However, a compensation in the capacity region allows for a leakage tolerance. In other words, rather than showing the cut-set bound is tight, we show that the cut-set upper bound is achievable to within a constant. Finally, using an appropriate lattice code, the third challenge is resolvable, too. In a lattice structure, the superposition of every two codewords is also a lattice codeword and therefore can be decoded at the relay. These will be addressed in the sections that follow.

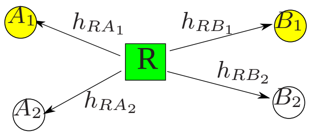
IV. Two-Pair Two Way Gaussian Relay Network

In this section we analyze the capacity region of the two-pair bidirectional Gaussian relay network shown in Figure 6. In particular, we show that the transmission scheme which was motivated in the previous section achieves within 2 bits/sec/Hz per user of the cut-set upper bound on the capacity region.

We consider two single-antenna transceiver pairs, (A_1, B_1) and (A_2, B_2), communicating to each other by exploiting a relay R. The relay is operating in the full-duplex mode, i.e. it can listen and transmit at the same time. We use a complex AWGN channel model for all channels



(a) Uplink



(b) Downlink

Fig. 6. Two-Pair bidirectional full-duplex relay network

in this network. Hence, the received signals at the nodes are given by

$$y_R = h_{A_1R}x_{A_1} + h_{B_1R}x_{B_1} + h_{A_2R}x_{A_2} + h_{B_2R}x_{B_2} + z_R,$$

$$y_{A_i} = h_{RA_i}x_R + z_{A_i}\ , \quad y_{B_i} = h_{RB_i}x_R + z_{B_i}, \qquad i = 1, 2$$

where x_{A_1}, x_{B_1}, x_{A_2}, x_{B_2}, and x_R are the signals transmitted from nodes A_1, B_1, A_2, B_2, and R, respectively. The transmit power constraint is $\mathbb{E}\left[|x_{A_i}|^2\right] = \mathbb{E}\left[|x_{B_i}|^2\right] = \mathbb{E}\left[|x_R|^2\right] \leq 1$ and the noises z_{A_1}, z_{B_1}, z_{A_2}, z_{B_2}, and z_R are all distributed as $\mathcal{CN}(0, 1)$. Note that the uplink channels gains (h_{A_iR} and h_{B_iR}) are not necessarily equal to the down-link channel gains (h_{RA_i} and h_{RB_i}), i.e. channel reciprocity is not assumed. For each pair (A_i,B_i), R_{A_i} is the rate at which A_i transmit data to B_i and R_{B_i} is the transmission rate of B_i to A_i.

We now begin by describing the cut-set upper bound [20], denoted by $\bar{\mathcal{C}}$, on the capacity region of this network:

$$\overline{\mathcal{C}} = \Big\{(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2}) \in \mathbb{R}_+^4 :$$

$$R_{A_i} \leq \min\left(C\left(|h_{A_iR}|^2\right), C\left(|h_{RB_i}|^2\right)\right) \tag{11}$$

$$R_{B_i} \leq \min\left(C\left(|h_{B_iR}|^2\right), C\left(|h_{RA_i}|^2\right)\right) \tag{12}$$

$$R_{A_1} + R_{A_2} \leq \min\Big(C\left(|h_{A_1R}|^2 + |h_{A_2R}|^2\right), C\left(\max\left(|h_{RB_1}|^2, |h_{RB_2}|^2\right)\right)\Big) \tag{13}$$

$$R_{B_1} + R_{B_2} \leq \min\Big(C\left(|h_{B_1R}|^2 + |h_{B_2R}|^2\right), C\left(\max\left(|h_{RA_1}|^2, |h_{RA_2}|^2\right)\right)\Big) \tag{14}$$

$$R_{A_1} + R_{B_2} \leq \min\Big(C\left(|h_{A_1R}|^2 + |h_{B_2R}|^2\right), C\left(\max\left(|h_{RB_1}|^2, |h_{RA_2}|^2\right)\right)\Big) \tag{15}$$

$$R_{B_1} + R_{A_2} \leq \min\Big(C\left(|h_{B_1R}|^2 + |h_{A_2R}|^2\right), C\left(\max\left(|h_{RA_1}|^2, |h_{RB_2}|^2\right)\right)\Big)\Big\}, \tag{16}$$

where $C(x) = \log(1 + x)$. The terms in (11)- (16) correspond to the cuts labeled from 1 to 8 in Fig. 7. Next, we define the up-link and down-link cut-set regions. The up-link cut-set region, $\mathcal{C}_u$, is the set of rates satisfying equations (11)-(16) when the down-link channel gains are assumed infinity. This means that the only restricting factors in determining the capacity regions are assumed to be the up-link channel gains. Likewise, the down-link cut-set region, $\mathcal{C}_d$, is the set of rates satisfying (11)-(16) in which the up-link channel gains are set to infinity. Note that $\bar{\mathcal{C}} = \mathcal{C}_d \cap \mathcal{C}_u$.

We say that a 4-tuple $(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2})$ is achievable if simultaneously A_i can communicate to B_i at rate R_{A_i} and B_i can communicate to A_i at rate R_{B_i} with arbitrary small error probability. The union of all achievable rate tuples is defined as the capacity region. We are now ready to state our main result.

Theorem 2: The capacity region of the two pair full-duplex bidirectional relay network is within 2 bits/sec/Hz per user of its cut-set upper bound described in (11)-(16). Or, more precisely, if

$$(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2}) \in \overline{\mathcal{C}}$$

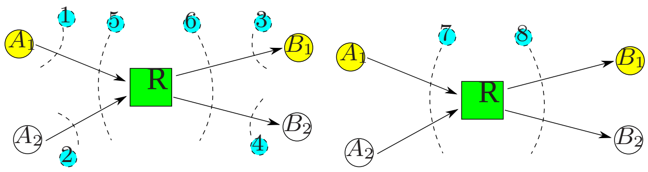
and $R_{A_i}, R_{B_i} \geq 2$ for $i = 1, 2$, then the rate tuple $(R_{A_1} - 2, R_{B_1} - 2, R_{A_2} - 2, R_{B_2} - 2)$ is achievable.

The rest of this section is devoted to proving this Theorem. First, we state the following lemma which helps us by limiting the number of rate configurations that we have to consider.

Lemma 2: Let $\mathbf{R} = (R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2})$ be a rate tuple in the cut-set region $\overline{\mathcal{C}}$. Assume $R_{A_i} \geq R_{B_i}$, $i = 1, 2$. Then it is always possible to sufficiently reduce the transmit powers at the uplink and add extra noise to the received signals at the downlink, such that new effective channel gains satisfy $|\tilde{h}_{A_iR}| \geq |\tilde{h}_{B_iR}|$ and $|\tilde{h}_{RB_i}| \geq |\tilde{h}_{RA_i}|$ for $i = 1, 2$, and $\mathbf{R}$ is still in the shrunk cut-set region.

Proof: See Appendix B. ■

This lemma basically reduces the number of relevant channel gain orderings that we have to consider in order to prove Theorem 2. Assume that the rate tuple that we want to show to be achievable (within 2 bits per user) satisfies $R_{A_i} \geq R_{B_i}$ for $i = 1, 2$. By Lemma 2, we can without loss of generality (wlog) assume that $|h_{A_iR}| \geq |h_{B_iR}|$ for $i = 1, 2$. We can also wlog assume that $|h_{A_1R}| \geq |h_{A_2R}|$ (otherwise we can re-label pair 1 and pair 2). Therefore, we only need to consider three different channel gain orderings for the uplink. Those three cases are shown in Fig. 8(a), 8(b) and 8(c). Similarly, we only need to consider three cases for the



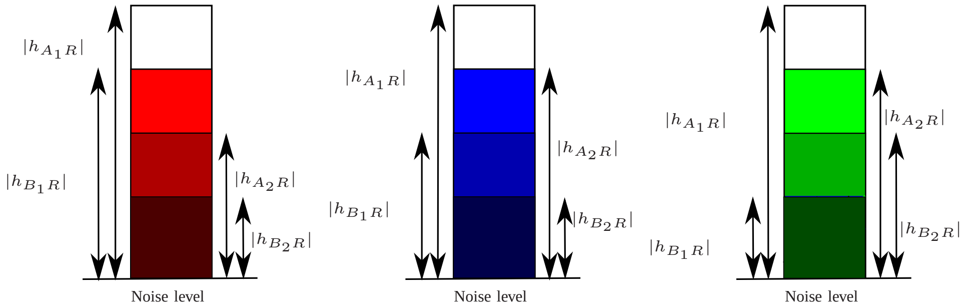
(a) Cuts (I) (b) Cuts (II)

Fig. 7. Cuts for the upper bound on the capacity region

downlink. To prove Theorem 2, first we describe the encoding strategy at the transmission nodes. As mentioned earlier, the idea is that strong transmitters of each pair split their signals into a Gaussian codeword and a lattice codeword, while the weak user only transmits a lattice codeword. While stating this encoding strategy we leave the power allocation parameters unspecified. In other words, the power level at which the user breaks up its message into the superposition of Gaussian and a lattice codeword remains as parameters. In the next step we mention the decoding at the relay where the superposition of lattice points and the Gaussian codewords are decoded. Afterwards, the relay maps each of the four decoded codewords into a random Gaussian codeword, and broadcasts their weighted superposition to all users. The last step is the decoding at the nodes, where every receiver first decodes the undesired codewords that have larger weights than the desired codewords. Thus, those codewords are decoded and successively canceled from the received signal one by one. Afterwards, both the weak and the strong receivers of each pair decode the Gaussian codeword corresponding to the lattice codeword belonging to that pair. In addition to that, the strong receivers decode one more codeword. This codeword corresponds to the Gaussian codeword, which was received by the relay from their transmitting strong counterpart. Eventually as a result of this scheme the rates that the users will successfully transmit will be a function of the power parameters that we set at the beginning. We will finally show that by choosing these parameters appropriately any rate tuple within 2 bits per user of the cut set is achievable.

A. Lattice Coding

In the following, some preliminaries and results on lattice coding are provided that we use in the remainder of the paper. We refer the interested reader to [21] for more details.



(a) Case I (b) Case II (c) Case III

Fig. 8. Three relevant configurations for the uplink and their corresponding received signal at the relay. At the lowest level, all signals are superposed, while at the next level (medium shade), all but one signals are superposed. At the top level (white) only one signal remains.

A lattice Λ of dimension n is described by

$$\mathbf{\Lambda} = \{\lambda = \mathbf{G}\mathbf{x} : \mathbf{x} \in \mathbb{Z}^n\}, \tag{17}$$

where $\mathbf{G}$ describes the lattice and is referred to as the generator matrix. The fundamental Voronoi region of such a lattice $\mathbf{\Lambda}$ is denoted by Ω. Furthermore, the volume of Ω, i.e. the reciprocical of the number of lattice point per unit volume, is denoted by V. Now, let p a positive integer and $\mathbb{Z}_p$ the set of integers modulo p. Further, let $\bar{v} : \mathbb{Z}^n \rightarrow \mathbb{Z}_p^n$ be the componentwise modulo operation over integer vectors. The lattices used in this paper are mod-p lattices, i.e. of the form

$$\Lambda_c = \{v \in \mathbb{Z}^n : \bar{v} \in C\}, \tag{18}$$

where C be a linear (n, k) code over $\mathbb{Z}_p$ and p is prime[21, Construction A]. Now, let $\mathcal{B}$ be a balanced set [21], [22] of linear (n, k) codes over $\mathbb{Z}_p$ and let $\mathcal{L}_{\mathcal{B}}$ be the set of lattices denoted by

$$\mathcal{L}_{\mathcal{B}} = \{\Lambda_c : C \in \mathcal{B}\}. \tag{19}$$

With this in mind, lets consider the following system model

$$y = x + z, \tag{20}$$

where y is the receive signal, x is the transmit signal and z is additive noise with zero mean and a variance σ^2. It was shown in [21, Theorem 4] that if the transmitted codeword is a lattice point, then there exists a lattice for that channel and the average probability of error with lattice decoding can be made arbitrarily small as the dimension of the lattice increases. Similarly, it was shown in [21] that by using a codebook $(\mathbf{\Lambda}+\mathbf{s})\cap\mathbf{S}$, where $\mathbf{s}$ is a shift and $\mathbf{S}$ describes the shaping gain, a rate R with arbitrarily small probability of error can be achieved if

$$R \leq \log\left(\frac{P}{\sigma^2}\right). \tag{21}$$

We will use this result in the remainder of the paper for the characterization of the rate region achievable with our proposed scheme.

B. *Encoding at the nodes*

Wlog assume that $R_{A_iR} \geq R_{B_iR}$. By Lemma 2 this means that we can assume $|h_{A_iR}| \geq |h_{B_iR}|$ and $|h_{RB_i}| \geq |h_{RA_i}|$. Then, the transmit signals at the nodes are given by

$$\begin{aligned} x_{A_i} &= \sqrt{\alpha_{A_i}^{(1)}}x_{A_i}^{(1)} + \sqrt{\alpha_{A_i}^{(2)}}x_{A_i}^{(2)}\ ,\ x_{B_i} = \sqrt{\alpha_{B_i}^{(2)}}x_{B_i}^{(2)} \quad i=1,2 \\ x_R &= \sum_{j=1}^{4}\sqrt{\alpha_R^{(j)}}x_R^{(j)} \qquad \text{with } \sum_{j}\alpha_R^{(j)} = 1, \end{aligned} \tag{22}$$

where $x_{A_i}^{(1)}$ and $x_R^{(j)}$ are codewords chosen from a random Gaussian codebook of size $2^{nR_{A_i}^{(1)}}$, $i=1,2$, and $2^{nR_R^{(j)}}$, for $j=1,\ldots,4$, respectively. $\mathbf{x}_{A_i}^{(2)}$ and $\mathbf{x}_{B_i}^{(2)}$, $i=1,2$, are lattice coded [21] using lattice ensembles $\{\Lambda_{A_1^{(2)}},\Lambda_{A_1^{(2)}},\Lambda_{B_1^{(2)}},\Lambda_{B_2^{(2)}}\}$ giving a codebook of size $2^{nR_{A_i}^{(2)}}$ and $2^{nR_{B_i}^{(2)}}$ with $i=1,2$, respectively. We assume that the second moment per dimension of the fundamental Voronoi region [21] of each lattice is $1/2$ which ensures satisfying the power constraint. At nodes A_i we have two messages $m_{A_i}^{(1)}$ and $m_{A_i}^{(2)}$ of size $2^{nR_{A_i}^{(1)}}$ and $2^{nR_{A_i}^{(2)}}$ that are mapped to $x_{A_i}^{(1)}$ and $x_{A_i}^{(2)}$, respectively. In other words, the strong transmitter of each pair transmits a superposition of a lattice code and a random Gaussian code, while the weaker user only transmits a lattice code. Thus, the transmit signals of nodes B_1 and B_2 reduce to

$$\begin{aligned} x_{B_1} &= \sqrt{\alpha_{B_1}^{(2)}}x_{B_1}^{(2)} \\ x_{B_2} &= \sqrt{\alpha_{B_2}^{(2)}}x_{B_2}^{(2)}. \end{aligned}$$

For the nodes A_1 and A_2, we have a superposition code (cf. (22)). Note that

$$t = x_{A_1}^{(2)} + x_{B_1}^{(2)} \ \text{ and } f = x_{A_2}^{(2)} + x_{B_2}^{(2)},$$

where t and f are also lattice points due to the group structure of the lattice [10].

The power parameters (i.e. α_{A_i} and α_{B_i}) are assigned such that the lattice codes of each pair arrive at the same power level to achieve signal scale alignment, so that the relay can decode the sum codeword correctly. Thus we set,

$$\alpha_{A_i}^{(2)} = \frac{|h_{B_iR}|^2}{|h_{A_iR}|^2}\alpha_{B_i}^{(2)}. \tag{23}$$

Furthermore, we should have $\alpha_{A_i}^{(1)} + \alpha_{A_i}^{(2)} \leq 1$ and $\alpha_{B_i}^{(2)} \leq 1$.

C. Uplink: Decoding at the relay

Recall that as discussed in Section III and illustrated in Figure 8 we have to analyze three cases only. Here, the analysis for the first case (cf. Fig. 8(a)) is given in detail. For the other cases, only the results are presented, since the other cases are similar and therefore omitted. However, along the presentation of the results, we also mention the differences should there be any.

1) Case $|h_{A_1R}| \geq |h_{B_1R}| \geq |h_{A_2R}| \geq |h_{B_2R}|$*:*

In the first case we have $|h_{A_1R}| \geq |h_{B_1R}| \geq |h_{A_2R}| \geq |h_{B_2R}|$.

The decoding order at the relay is as follows. First the relay decodes the Gaussian $x_{A_1}^{(1)}$, then the lattice point t from A_1 and B_1, followed by $x_{A_2}^{(1)}$ and finally the lattice point f from A_2 and B_2. We can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(1)} \leq C\left(\frac{|h_{A_1R}|^2\alpha_{A_1}^{(1)}}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + \alpha_{A_2}^{(1)}|h_{A_2R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{24}$$

$$R_{A_1}^{(2)}, R_{B_1} \leq \log\left(\frac{|h_{B_1R}|^2\alpha_{B_1}^{(2)}}{\alpha_{A_2}^{(1)}|h_{A_2R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right)^+ \tag{25}$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \left(\log\left(\alpha_{B_2}^{(2)}|h_{B_2R}|^2\right)\right)^+, \quad R_{A_2}^{(1)} \leq C\left(\frac{|h_{A_2R}|^2\alpha_{A_2}^{(1)}}{2|h_{B_2R}|^2\alpha_{B_2}^{(2)} + 1}\right). \tag{26}$$

Details of the derivations are given in Appendix C.

2) Case $|h_{A_1R}| \geq |h_{A_2R}| \geq |h_{B_1R}| \geq |h_{B_2R}|$*:* The decoding order at the relay is as follows. First the relay decodes the Gaussian $x_{A_1}^{(1)}$ and $x_{A_2}^{(1)}$ simultaneously by treating the remaining signals as noise. Afterwards, the lattice point t from A_1 and B_1 is decoded, followed by the lattice point f from A_2 and B_2. We can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(1)} \leq C\left(\frac{|h_{A_1R}|^2\alpha_{A_1}^{(1)}}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{27}$$

$$R_{A_2}^{(1)} \leq C\left(\frac{\alpha_{A_2}^{(1)}|h_{A_2R}|^2}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{28}$$

$$R_{A_1}^{(1)} + R_{A_2}^{(1)} \leq C\left(\frac{|h_{A_1R}|^2\alpha_{A_1}^{(1)} + \alpha_{A_2}^{(1)}|h_{A_2R}|^2}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{29}$$

$$R_{A_1}^{(2)}, R_{B_1} \leq \log\left(\frac{|h_{B_1R}|^2\alpha_{B_1}^{(2)}}{2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right)^+ \tag{30}$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \left(\log\left(\alpha_{B_2}^{(2)}|h_{B_2R}|^2\right)\right)^+. \tag{31}$$

3) Case $|h_{A_1R}| \geq |h_{A_2R}| \geq |h_{B_2R}| \geq |h_{B_1R}|$*:*

The decoding is similar to the above case, except that the lattice point f from A_2 and B_2 is decoded before decoding the lattice point t from A_1 and B_1 . Again, we can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(1)} \leq C\left(\frac{|h_{A_1R}|^2\alpha_{A_1}^{(1)}}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{32}$$

$$R_{A_2}^{(1)} \leq C\left(\frac{\alpha_{A_2}^{(1)}|h_{A_2R}|^2}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{33}$$

$$R_{A_1}^{(1)} + R_{A_2}^{(1)} \leq C\left(\frac{|h_{A_1R}|^2\alpha_{A_1}^{(1)} + \alpha_{A_2}^{(1)}|h_{A_2R}|^2}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right) \tag{34}$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \log\left(\frac{|h_{B_2R}|^2\alpha_{B_2}^{(2)}}{2\alpha_{B_1}^{(2)}|h_{B_1R}|^2 + 1}\right)^+ \tag{35}$$

$$R_{A_1}^{(2)}, R_{B_1} \leq \left(\log\left(\alpha_{B_1}^{(2)}|h_{B_1R}|^2\right)\right)^+. \tag{36}$$

Now we state the following lemma whose proof is given in Appendix D.

Lemma 3: Suppose that the nodes are using the transmit strategy described in Section IV-B. Then for any 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying

$$r_{A_1} \leq C\left(|h_{A_1R}|^2\right) - 2 \,, r_{B_1} \leq C\left(|h_{B_1R}|^2\right) - 1 \tag{37}$$

$$r_{A_2} \leq C\left(|h_{A_2R}|^2\right) - 2 \,, r_{B_2} \leq C\left(|h_{B_2R}|^2\right) - 1 \tag{38}$$

$$r_{A_1} + r_{A_2} \leq C\left(|h_{A_1R}|^2 + |h_{A_2R}|^2\right) - 4 \tag{39}$$

$$r_{A_1} + r_{B_2} \leq C\left(|h_{A_1R}|^2 + |h_{B_2R}|^2\right) - 4 \tag{40}$$

$$r_{B_1} + r_{B_2} \leq C\left(|h_{B_1R}|^2 + |h_{B_2R}|^2\right) - 4 \tag{41}$$

$$r_{B_1} + r_{A_2} \leq C\left(|h_{B_1R}|^2 + |h_{A_2R}|^2\right) - 4, \tag{42}$$

there exists a choice of power assignments ($\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$) such that the relay can use the decoding strategy described earlier to decode the Gaussian $x_{A_i}^{(1)}$ of rate $R_{A_i}^{(1)} = r_{A_i} - r_{B_i}$, the lattice point t of rate $R_{A_1}^{(2)} = R_{B_1} = r_{B_1}$, and the lattice point f of rate $R_{A_2}^{(2)} = R_{B_2} = r_{B_2}$, with arbitrary small error probability.

D. Encoding at the relay

The relay maps the decoded $x_{A_1}^{(1)}$, t, $x_{A_2}^{(1)}$, and f to a Gaussian codeword $x_R^{(1)}$ of size $2^{nR_{A_1}^{(1)}}$, $x_R^{(2)}$ of size $2^{nR_{B_1}}$, $x_R^{(3)}$ of size $2^{nR_{A_2}^{(1)}}$, and $x_R^{(4)}$ of size $2^{nR_{B_2}}$, respectively.

E. Downlink: Decoding at the nodes

As in the uplink, we have to consider three cases only, from which we provide the detailed analysis for $|h_{RB_1}| \geq |h_{RA_1}| \geq |h_{RB_2}| \geq |h_{RA_2}|$. The other cases follow similar lines of arguments and thus only the results are presented.

The relay uses a superposition of four messages. One message is decoded by all users. Another message is decoded by both users of the first pair and the strong receiver of the second pair. Yet another message is decoded by only the strong receiver of the first pair, and finally the remaining message is decoded by both users of the first pair.

1) Case $|h_{RB_1}| \geq |h_{RA_1}| \geq |h_{RB_2}| \geq |h_{RA_2}|$: We can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(2)}, R_{B_1} \leq \min\left(C\left(\frac{|h_{RB_1}|^2\alpha_R^{(2)}}{1+|h_{RB_1}|^2\alpha_R^{(1)}}\right), C\left(|h_{RA_1}|^2\alpha_R^{(2)}\right)\right), \tag{43}$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \min\left(C\left(\frac{|h_{RB_2}|^2\alpha_R^{(4)}}{1+|h_{RB_2}|^2\sum_{j=1}^{3}\alpha_R^{(j)}}\right),\right.$$

$$\left. C\left(\frac{|h_{RA_2}|^2\alpha_R^{(4)}}{1+|h_{RA_2}|^2\left(\alpha_R^{(1)}+\alpha_R^{(2)}\right)}\right)\right), \tag{44}$$

$$R_{A_1}^{(1)} \leq C\left(|h_{RB_1}|^2\alpha_R^{(1)}\right), \quad R_{A_2}^{(1)} \leq C\left(\frac{|h_{RB_2}|^2\alpha_R^{(3)}}{1+|h_{RB_2}|^2\left(\alpha_R^{(1)}+\alpha_R^{(2)}\right)}\right). \tag{45}$$

Details of the derivation are given in Appendix E.

2) Case $|h_{RB_1}| \geq |h_{RB_2}| \geq |h_{RA_1}| \geq |h_{RA_2}|$: We can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(2)}, R_{B_1} \leq \min\left(C\left(\frac{|h_{RA_1}|^2\alpha_R^{(2)}}{1+|h_{RA_1}|^2\alpha_R^{(3)}}\right), C\left(\frac{|h_{RB_2}|^2\alpha_R^{(2)}}{1+|h_{RB_2}|^2\left(\alpha_R^{(1)}+\alpha_R^{(3)}\right)}\right)\right), \tag{46}$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \min\left(C\left(\frac{|h_{RB_2}|^2\alpha_R^{(4)}}{1+|h_{RB_2}|^2\sum_{j=1}^{3}\alpha_R^{(j)}}\right),\right.$$

$$\left. C\left(\frac{|h_{RA_1}|^2\alpha_R^{(4)}}{1+|h_{RA_1}|^2\left(\alpha_R^{(2)}+\alpha_R^{(3)}\right)}\right), C\left(\frac{|h_{RA_2}|^2\alpha_R^{(4)}}{1+|h_{RA_2}|^2\left(\alpha_R^{(1)}+\alpha_R^{(2)}\right)}\right)\right), \tag{47}$$

$$R_{A_1}^{(1)} \leq C\left(|h_{RB_1}|^2\alpha_R^{(1)}\right), \quad R_{A_2}^{(1)} \leq C\left(\frac{|h_{RB_2}|^2\alpha_R^{(3)}}{1+|h_{RB_2}|^2\alpha_R^{(1)}}\right). \tag{48}$$

3) Case $|h_{RB_1}| \geq |h_{RB_2}| \geq |h_{RA_2}| \geq |h_{RA_1}|$: We can show that for any choice of $\alpha_{A_i}^{(j)}$ and $\alpha_{B_i}^{(2)}$, this can be done successfully as long as,

$$R_{A_1}^{(2)}, R_{B_1} \leq \min\left(C\left(\frac{|h_{RB_2}|^2\alpha_R^{(2)}}{1+|h_{RB_2}|^2\sum_{j=1,j\neq 2}^4 \alpha_R^{(j)}}\right),\right.$$

$$\left. C\left(\frac{|h_{RA_1}|^2\alpha_R^{(2)}}{1+|h_{RA_1}|^2\left(\alpha_R^{(4)}+\alpha_R^{(3)}\right)}\right), C\left(\frac{|h_{RA_2}|^2\alpha_R^{(2)}}{1+|h_{RA_2}|^2\left(\alpha_R^{(1)}+\alpha_R^{(4)}\right)}\right)\right), \tag{49}$$

$$R_{A_2}^{(2)}, R_{B_2} \leq \min\left(C\left(\frac{|h_{RA_2}|^2\alpha_R^{(4)}}{1+|h_{RA_2}|^2\alpha_R^{(1)}}\right), C\left(\frac{|h_{RB_2}|^2\alpha_R^{(4)}}{1+|h_{RB_2}|^2\left(\alpha_R^{(1)}+\alpha_R^{(3)}\right)}\right)\right), \tag{50}$$

$$R_{A_1}^{(1)} \leq C\left(|h_{RB_1}|^2\alpha_R^{(1)}\right), \quad R_{A_2}^{(1)} \leq C\left(\frac{|h_{RB_2}|^2\alpha_R^{(3)}}{1+|h_{RB_2}|^2\alpha_R^{(1)}}\right). \tag{51}$$

Now we state the following lemma whose proof is given in Appendix F.

Lemma 4: Suppose that the relay is using the transmit strategy described above. Then for any 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying

$$r_{A_1} \leq C\left(|h_{RB_1}|^2\right) - 2\ , r_{B_1} \leq C\left(|h_{RA_1}|^2\right) - 2 \tag{52}$$

$$r_{A_2} \leq C\left(|h_{RB_2}|^2\right) - 2\ , r_{B_2} \leq C\left(|h_{RA_2}|^2\right) - 2 \tag{53}$$

$$r_{A_1} + r_{A_2} \leq C\left(\max\left(|h_{RB_1}|^2, |h_{RB_2}|^2\right)\right) - 3 \tag{54}$$

$$r_{A_1} + r_{B_2} \leq C\left(\max\left(|h_{RB_1}|^2, |h_{RA_2}|^2\right)\right) - 3 \tag{55}$$

$$r_{B_1} + r_{B_2} \leq C\left(\max\left(|h_{RA_1}|^2, |h_{RA_2}|^2\right)\right) - 3 \tag{56}$$

$$r_{B_1} + r_{A_2} \leq C\left(\max\left(|h_{RA_1}, |h_{RB_2}|^2\right)\right) - 3 \tag{57}$$

there exists a choice of power assignments ($\alpha_R^{(j)}$'s) such that B_1 can decode the Gaussian codewords $x_R^{(1)}$ of rate $R_{A_1}^{(1)} = r_{A_1} - r_{B_1}$, A_1 and B_1 can both decode the Gaussian codeword $x_R^{(2)}$ of rate $R_{A_1}^{(2)} = R_{B_1} = r_{B_1}$, B_2 can decode the Gaussian codeword $x_R^{(3)}$ of rate $R_R^{(3)} = r_{A_2} - r_{B_2}$, and A_2 and B_2 can both decode the Gaussian codeword $x_R^{(4)}$ of rate $R_{A_2}^{(2)} = R_{B_2} = r_{B_2}$, with arbitrary small error probability.

Now note that if

$$(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2}) \in \overline{\mathcal{C}}$$

and $R_{A_i}, R_{B_i} \geq 2$ for $i = 1, 2$, then the rate tuple

$$(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2}) = (R_{A_1} - 2, R_{B_1} - 2, R_{A_2} - 2, R_{B_2} - 2)$$

satisfies the conditions of both Lemma 3 and 4. Therefore by the proposed strategy the rate tuple $(R_{A_1}-2, R_{B_1}-2, R_{A_2}-2, R_{B_2}-2)$ is achievable, and this completes the proof of Theorem 2.

V. Conclusion

In this paper we studied the multi-pair bidirectional relay network which is a generalization of the bidirectional relay channel. We examined this problem in the context of the deterministic channel model introduced in [14] and characterized its capacity region completely in both full-duplex and half-duplex cases. We also showed that the capacity can be achieved by a divide-and-conquer strategy with a simple equation-forwarding scheme at the relay and illustrated some structures on the signal levels that these equations are created at. Based on insights gained from the deterministic channel model, we proposed a transmission strategy for the Gaussian two-pair two-way full-duplex relay network and found an approximate characterization of the capacity region. In fact, we proposed a specific superposition coding scheme that achieves to within 2 bits per user of the cut-set upper bound on the capacity of the two-pair two-way relay network. Possible directions for future work is the extension to the half-duplex mode. Extension of the proposed transmission strategy to the case that there are more than two pairs is possible, however, analyzing the gap between the achievable rate of the corresponding scheme and the cut-set upper bound is expected to be quite cumbersome.

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Appendix A
Proof of Lemma 1

Proof: In the following, we only prove the lemma for $M = 2$ pairs, but the proof can be easily generalized to any arbitrary number of pairs. We use induction on the sum rate to show that every integer 4-tuple $(R_{A_1}, R_{B_1}, R_{A_2}, R_{B_2})$ satisfying the cut set bound is achievable by allocating subsets of the signal levels exclusively to users of different sessions, and using equation forwarding at the relay. For convenience we consider two separate cases, and mention how in each case we can designate one relay signal level in uplink and one signal level in down-link for serving one of the sessions as many as 1 bits per user. We then show that the reduced rate tuple is within the cut-set region of the reduced network. By reduced network, we mean the new network after eliminating the designated signal levels. We use an example along the proof to illustrate the ideas. The example network is shown in Figure 5, and one can check that the rate-tuple (3,1,2,1) is inside the cut-set region associated with this network.

Case 1: There is a pair where both nodes have nonzero transmission rates. Without loss of generality we may assume that R_{A_1} and R_{B_1} are both nonzero. Our goal is to choose one up-link signal level and one down-link signal level, and assign them to the (A_1,B_1) session. A_1 and B_1 will then transmit one bit at the specified uplink level to the relay, and the relay will transmit (broadcast) the received equation at the specified down-link level to both A_1 and B_1. After doing so and removing the specified signal levels, the network will reduce to an equivalent network with lower capacities. However, we also show that the new rates satisfy the cut-set bounds of (2) to (9) for the new network, and therefore by induction we can say that the cut-set bound is achievable in this case.

We claim that the appropriate up-link signal level is *the highest signal level connected to both* A_1 *and* B_1, and the appropriate down-link signal level is the *lowest signal level connected to both* A_1 *and* B_1. We refer to these levels by l_u and l_d respectively. For example, in the network of Figure 5, l_u and l_d correspond to the third relay's circle from the bottom in UL, and third relay's circle from the top in DL respectively. Now, we claim that after removing l_u and l_d from UL and DL, the new rate tuple $(R_{A_1}-1, R_{B_1}-1, R_{A_2}, R_{B_2})$ satisfies the cut-set bounds of the new network. First, note that by removing the levels l_u and l_d, each of the values n_{A_1R}, n_{B_1R}, n_{RA_1} and n_{RB_1} will decrease by exactly 1, and the other four capacities n_{A_2R}, n_{B_2R}, n_{RA_2} and n_{RB_2} will decrease by at most 1, regardless of their original values. So, after removing l_u and l_d and reducing R_{A_1} and R_{B_1} by 1, inequalities (2) and (3) clearly continue to hold . Besides, (6) to (9) remain valid too, since in each of them the L.H.S is decreased by 1 whereas the R.H.S is deceased by at most 1. Because of the symmetry, among the remaining equations, we only have to show that inequality (4) remains valid. Let's say (4) is violated because of eliminating the uplink signal level l_u. This means that originally we should have had:

$$R_{A_2} \leq n_{A_2R} \tag{58}$$

On the other hand l_u is connected to A_2. Because we chose l_u to be the highest level in up-link that is connected to both A_1 and B_1, the inequality in (58) implies that $n_{A_2R} \geq \min\{n_{A_1R}, n_{B_1R}\}$. Let's assume $n_{A_2R} \geq n_{A_1R}$. This and (6) imply that $n_{A_2R} \geq R_{A_1} + R_{A_2}$. But we already have $R_{A_2} = n_{A_2R}$, which means $R_{A_1} = 0$, which contradicts our initial assumption of nonzero rates for the pair (A_1,B_1). Similarly if $n_{A_2R} \geq n_{B_1R}$, from (9) we conclude $n_{A_2R} \geq R_{B_1} + R_{A_2}$. This and the fact that $R_{A_2} = n_{A_2R}$ imply $R_{B_1} = 0$, which again cannot be true because of the original assumption.

In a similar way we can show that the removal of l_d cannot violate the validity of (4). The reason is if $R_{A_2} = n_{RB_2}$ and l_d is in the transmission range of relay to B_2 (i.e. the circle corresponding to level l_d in Figure 5 is connected to B_2) then $n_{RB_2} \geq \min\{n_{RA_1}, n_{RB_1}\}$. Now if $n_{RB_2} \geq n_{RA_1}$ then from (9) we obtain $n_{RB_2} \geq R_{A_2} + R_{B_1}$. But $R_{A_2} = n_{RB_2}$ which means $R_{B_1} = 0$. Likewise, if $n_{RB_2} \geq n_{RB_1}$ from equation (6) we conclude $n_{RB_2} \geq R_{A_1} + R_{A_2}$, which implies $R_{A_1} = 0$, which again is a contradiction.

Let's apply this scheme to the example network. We should first take the (A_1,B_1) pair and serve them through one signal level in UL and one level in DL. Next, we take the (A_2,B_2) and similarly assign corresponding levels in UL and DL to them. These two steps are shown in Figures 4(a) and 4(b). For the sake of clarity, the removed signal levels are dotted in each step. The remaining unserved rates are (2,0,1,0).

Case 2: Every session has a node with zero rate. Without loss of generality (wlog), say $R_{B_1} = R_{B_2} = 0$ (only A_1 and A_2 want to communicate with their end parties). We show that in this case A_1 and A_2 can respectively transmit R_{A_1} and R_{A_2} bits at different signal levels to the relay, and the relay can forward those bits to B_1 and B_2 by using distinct signal levels in down-link, without any interference. In other words, the relay can *rearrange* the received equations and forward them. In fact, if $n_{A_1R} \leq n_{A_2R}$, then A_1 can use the lowest R_{A_1} signal levels in up-link to transmit R_{A_1} bits to the relay, and A_2 can use another R_{A_2} levels on top of those used by A_1 to transmit R_{A_2} bits to the relay. This is feasible because of the inequalities in (2) and (6). However, if $n_{A_2R} \leq n_{A_1R}$, A_2 can use the lowest R_{A_2} signal levels and A_1 is able to transmit his bits on signals levels above those used by A_2. Similarly, in DL if $n_{RB_1} \leq n_{RB_2}$ the bits addressed to B_1 will be put on the highest signal levels and the bits addressed to B_2 are placed on the signal levels below those of B_1. This means that independent of the channel capacities in the up-link, the relay can always re-order the received equations and transmit and forward them on distinct signal levels to the destinations . Figure 4(c) shows how this idea is applied to our example network. The final configuration that achieves the rate-tuple for this example is shown in Figure 5. ■

APPENDIX B
PROOF OF LEMMA 2

Since the proof for both pairs are similar, we only bring the proof for pair $i = 1$. We claim that if $|h_{B_1R}| > |h_{A_1R}|$ and $\mathbf{R} \in \mathcal{C}_u$, then $\mathbf{R} \in \mathcal{C}'_u$, where $\mathcal{C}'_u$ is the up-link cut-set region of the network resulted by weakening $|h_{B_1R}|$ and setting it equal to $|h_{A_1R}|$. We call the new (undermined) uplink channel gains $(h'_{A_1R}, h'_{B_1R}, h'_{A_2R}, h'_{B_2R})$. The claim is justified by check marking equations (11) to (16) for new capacities (with infinite down-link channel gains). The only non-obvious inequalities are the ones in which h'_{B_1R} appears. By symmetry we only have to verify that (12) and (16) hold. Start with the original equations for $(h_{A_1R}, h_{B_1R}, h_{A_2R}, h_{B_2R})$ and note that the LHS of equations (12) and (16) are less than or equal to the LHS of (11) and (13) respectively and thus less than their RHS. Now replace h_{A_1R} with h'_{B_1R} and h_{A_2R} with h'_{A_2R} to get the desired inequalities. A similar argument on the down-link cut-set region shows that we can make the down-link channel gains of each pair consistent (in ordering) with the transmission rate and this completes the proof.

APPENDIX C
DECODING AT THE RELAY

We receive the following signal at the relay

$$\begin{aligned} y_R = {} & h_{A_1R}\sqrt{\alpha^{(1)}_{A_1}}x^{(1)}_{A_1} + h_{A_1R}\sqrt{\alpha^{(2)}_{A_1}}x^{(2)}_{A_1} + h_{B_1R}x_{B_1} \\ & + h_{A_2R}\sqrt{\alpha^{(1)}_{A_2}}x^{(1)}_{A_2} + h_{A_2R}\sqrt{\alpha^{(2)}_{A_2}}x^{(2)}_{A_2} + h_{B_2R}x_{B_2} + z_R. \end{aligned}$$

For the case considered here ($|h_{A_1R}| \geq |h_{B_1R}| \geq |h_{A_2R}| \geq |h_{B_2R}|$), we have the following decoding order at the relay: $x^{(1)}_{A_1} \to t \to x^{(1)}_{A_2} \to f$. It follows the decoding of the signals from pair (A_1, B_1).

Decoding of $x^{(1)}_{A_1}$ can be done with low error probability as long as

$$R^{(1)}_{A_1} \leq C\left(\frac{|h_{A_1R}|^2\alpha^{(1)}_{A_1}}{2\alpha^{(2)}_{B_1}|h_{B_1R}|^2 + \alpha^{(1)}_{A_2}|h_{A_2R}|^2 + 2\alpha^{(2)}_{B_2}|h_{B_2R}|^2 + 1}\right)$$

Once $x_{A_1}^{(1)}$ is decoded, it can be subtracted successfully from the received signal. Thus, we have

$$\tilde{y}_R = h_{B_1R}\sqrt{\alpha_{B_1}^{(2)}}\underbrace{\left(x_{A_1}^{(2)} + x_{B_1}^{(2)}\right)}_{t} + h_{A_2R}\sqrt{\alpha_{A_2}^{(1)}}x_{A_2}^{(1)} + h_{B_2R}\sqrt{\alpha_{B_2}^{(2)}}\underbrace{\left(x_{A_2}^{(2)} + x_{B_2}^{(2)}\right)}_{f} + z_R$$

Next, the sum codeword t of the lattice codes from $x_{A_1}^{(2)}$ and x_{B_1} is decoded. The decoding of t can be done with low error probability as long as

$$R_{A_1}^{(2)}, R_{B_1} \leq \log\left(\frac{|h_{B_1R}|^2\alpha_{B_1}^{(2)}}{\alpha_{A_2}^{(1)}|h_{A_2R}|^2 + 2\alpha_{B_2}^{(2)}|h_{B_2R}|^2 + 1}\right)^+.$$

Once t is decoded, it can be subtracted successfully from the received signal. Thus, we have

$$\hat{y}_R = h_{A_2R}\sqrt{\alpha_{A_2}^{(1)}}x_{A_2}^{(1)} + h_{B_2R}\sqrt{\alpha_{B_2}^{(2)}}f + z.$$

It follows the decoding of the signals from pair (A_2, B_2). beginning with the decoding of the Gaussian $x_{A_2}^{(1)}$. This can be done with low probability as long as

$$R_{A_2}^{(1)} \leq C\left(\frac{|h_{A_2R}|^2\alpha_{A_2}^{(1)}}{2|h_{B_2R}|^2\alpha_{B_2}^{(2)} + 1}\right).$$

Once $x_{A_2}^{(1)}$ is decoded, it can be subtracted successfully from the received signal. Thus, we have

$$\hat{y}_R = \sqrt{\alpha_{B_2}^{(2)}}h_{B_2R}f + z$$

As a final step, we want to decode the lattice point f. This can be done with low probability as long as

$$R_{B_2} \leq \left(\log\left(\alpha_{B_2}^{(2)}|h_{B_2R}|^2\right)\right)^+.$$

APPENDIX D
PROOF OF LEMMA 3

The three cases we have to consider are given in sections IV-C1 to IV-C3. In the following we provide the proof for each case separately.

A. *Case* $|h_{A_1R}| \geq |h_{B_1R}| \geq |h_{A_2R}| \geq |h_{B_2R}|$

Consider a 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying (37)-(42). Starting with (26), we equate

$$\left(\log\left(\alpha_{B_2}^{(2)}|h_{B_2R}|^2\right)\right)^+ = r_{B_2} \Rightarrow \alpha_{B_2}^{(2)} = \frac{2^{r_{B_2}}}{|h_{B_2R}|^2}. \tag{59}$$

Now from (38) we know that

$$\alpha_{B_2}^{(2)} \leq \frac{1+|h_{B_2R}|^2}{2|h_{B_2R}|^2} \overset{|h_{B_2R}|\geq 1}{\leq} 1, \tag{60}$$

which shows that this is a valid choice of $\alpha_{B_2}^{(2)}$. Next we equate $r_{A_2} - r_{B_2} =$ RHS of (26) and use (59). We get

$$\alpha_{A_2}^{(1)} = \frac{\left(2^{r_{A_2}-r_{B_2}}-1\right)\left(2^{r_{B_2}}+1\right)}{|h_{A_2R}|^2}. \tag{61}$$

Using (23) and adding this to (61) we get

$$\alpha_{A_2}^{(1)} + \alpha_{A_2}^{(2)} = \frac{22^{r_{A_2}} + 2^{r_{A_2}-r_{B_2}} - 2^{r_{B_2}} - 1}{|h_{A_2R}|^2} \leq \frac{32^{r_{A_2}} - 2}{|h_{A_2R}|^2} \overset{(38)}{\leq} 1, \tag{62}$$

verifying that this is a valid choice of $\alpha_{A_2}^{(1)}$, $\alpha_{A_2}^{(2)}$. Then we equate $r_{B_1} =$ RHS of (25), by setting

$$\alpha_{B_1}^{(2)} = \frac{2^{r_{B_1}}2^{r_{A_2}-r_{B_2}}\left(22^{r_{B_2}}+1\right)}{|h_{B_1R}|^2} \leq \frac{32^{r_{B_1}+r_{A_2}}}{|h_{B_1R}|^2} \overset{(42),|h_{B_1R}|^2\geq\frac{3}{2}}{\leq} 1, \tag{63}$$

verifying that this is a valid choice of $\alpha_{B_1}^{(2)}$. Finally we equate $r_{A_1} - r_{B_1} =$ RHS of (24), by setting

$$\alpha_{A_1}^{(2)} = \frac{\left(2^{r_{A_1}-r_{B_1}}-1\right)\left(2^{r_{A_2}+r_{B_1}-r_{B_2}}\left(1+22^{r_{B_2}}\right)+2^{r_{A_2}-r_{B_2}}\left(1+2^{r_{B_2}}\right)+2^{r_{B_2}}\right)}{|h_{A_1R}|^2}. \tag{64}$$

Using (23) and (63) and adding this to (64) we get

$$\alpha_{A_1}^{(1)} + \alpha_{A_1}^{(2)} \leq \frac{5\cdot 2^{r_{A_1}+r_{A_2}} + 2^{r_{A_1}+r_{B_2}} - 3}{|h_{A_1R}|^2} \overset{(39)}{\leq} 1.$$

which shows that this is a valid choice of $\alpha_{A_1}^{(1)}$, $\alpha_{A_1}^{(2)}$.

B. *Case* $|h_{A_1R}| \geq |h_{A_2R}| \geq |h_{B_1R}| \geq |h_{B_2R}|$

Consider a 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying (37)-(42). Starting with (31), we equate

$$\left(\log\left(\alpha_{B_2}^{(2)}|h_{B_2R}|^2\right)\right)^+ = r_{B_2} \Rightarrow \alpha_{B_2}^{(2)} = \frac{2^{r_{B_2}}}{|h_{B_2R}|^2}. \tag{65}$$

Now from (38) we know that

$$\alpha_{B_2}^{(2)} \leq \frac{1+|h_{B_2R}|^2}{2|h_{B_2R}|^2} \overset{|h_{B_2R}|\geq 1}{\leq} 1, \tag{66}$$

which shows that this is a valid choice of $\alpha_{B_2}^{(2)}$. Next we equate $r_{B_1} =$ RHS of (30), by setting

$$\alpha_{B_1}^{(2)} = \frac{2^{r_{B_1}}\left(22^{r_{B_2}}+1\right)}{|h_{B_1R}|^2} \overset{(37),(41),|h_{B_1R}|^2\geq 2}{\leq} 1, \tag{67}$$

verifying that this is a valid choice of $\alpha_{B_1}^{(2)}$. Then we equate $r_{A_2} - r_{B_2} =$ RHS of (28), by setting

$$\alpha_{A_2}^{(1)} = \frac{\left(2^{r_{A_2}-r_{B_2}}-1\right)\left(42^{r_{B_1}+r_{B_2}}+2\left(2^{r_{B_1}+1}+2^{r_{B_2}}\right)+1\right)}{|h_{A_2R}|^2}. \tag{68}$$

Using (23), $2^x + 2^y \leq 2^{x+y}$ with $x, y \geq 1$, and (65) and adding this to (68) we get

$$\alpha_{A_2}^{(1)} + \alpha_{A_2}^{(2)} \leq \frac{62^{r_{A_2}+r_{B_1}+r_{A_2}-8}}{|h_{A_2R}|^2} \overset{(39),(42)}{\leq} 1, \tag{69}$$

verifying that this is a valid choice of $\alpha_{A_2}^{(1)}$, $\alpha_{A_2}^{(2)}$. Now we equate $r_{A_1} - r_{B_1} =$ RHS of (27), by setting

$$\alpha_{A_1}^{(1)} = \frac{\left(2^{r_{A_1}-r_{B_1}}-1\right)\left(42^{r_{B_1}+r_{B_2}}+2\left(2^{r_{B_1}+1}+2^{r_{B_2}}\right)+1\right)}{|h_{A_1R}|^2}. \tag{70}$$

Using (23), $2^x + 2^y \leq 2^{x+y}$ with $x, y \geq 1$, and (67) and adding this to (70) we get

$$\begin{aligned}\alpha_{A_1}^{(1)} + \alpha_{A_1}^{(2)} &= \frac{\left(2^{r_{A_1}-r_{B_1}}-1\right)\left(42^{r_{B_1}+r_{B_2}}+2\left(2^{r_{B_1}+1}+2^{r_{B_2}}\right)+1\right)}{|h_{A_1R}|^2} + \frac{2^{r_{B_1}}\left(22^{r_{B_2}}+1\right)}{|h_{A_1R}|^2} \\ &\leq \frac{6\cdot 2^{r_{A_1}+r_{B_2}}+2^{r_{A_1}}-6}{|h_{A_1R}|^2} \overset{(37),(40)}{\leq} 1.\end{aligned}$$

which shows that this is a valid choice of $\alpha_{A_1}^{(1)}$, $\alpha_{A_1}^{(2)}$.

Finally we equate $r_{A_1} - r_{B_1} + r_{A_2} - r_{B_2} =$ RHS of (29), by setting

$$\alpha_{A_1}^{(1)} = \frac{\left(2^{r_{A_1}-r_{B_1}+r_{A_2}-r_{B_2}}-2^{r_{A_2}-r_{B_2}}\right)\left(42^{r_{B_2}+r_{B_1}}+2\left(2^{r_{B_2}}+2^{r_{B_1}}\right)+1\right)}{|h_{A_1R}|^2}. \tag{71}$$

Using (23), $2^x + 2^y \leq 2^{x+y}$ with $x, y \geq 1$, and (67) and adding this to (71) we get

$$\begin{aligned}\alpha_{A_1}^{(1)} + \alpha_{A_1}^{(2)} &= \left(2^{r_{A_1}-r_{B_1}+r_{A_2}-r_{B_2}}-2^{r_{A_2}-r_{B_2}}\right)\frac{42^{r_{B_2}+r_{B_1}}+2\left(2^{r_{B_2}}+2^{r_{B_1}}\right)+1}{|h_{A_1R}|^2} \\ &+ \frac{2^{r_{B_1}}\left(22^{r_{B_2}}+1\right)}{|h_{A_1R}|^2} \overset{r_{A_2}\geq r_{B_2}}{\leq} \frac{7\cdot 2^{r_{A_1}+r_{A_2}}-2}{|h_{A_1R}|^2} \overset{(39)}{\leq} 1.\end{aligned}$$

which shows that this is a valid choice of $\alpha_{A_1}^{(1)}$, $\alpha_{A_1}^{(2)}$.

C. *Case* $|h_{A_1R}| \geq |h_{A_2R}| \geq |h_{B_2R}| \geq |h_{B_1R}|$

Consider a 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying (37)-(42). Starting with (36), we equate

$$\left(\log\left(\alpha_{B_1}^{(2)}|h_{B_1R}|^2\right)\right)^+ = r_{B_1} \Rightarrow \alpha_{B_1}^{(2)} = \frac{2^{r_{B_1}}}{|h_{B_1R}|^2}. \tag{72}$$

Now from (37) we know that

$$\alpha_{B_1}^{(2)} \leq \frac{1+|h_{B_1R}|^2}{2|h_{B_1R}|^2} \overset{|h_{B_1R}|\geq 1}{\leq} 1, \tag{73}$$

which shows that this is a valid choice of $\alpha_{B_1}^{(2)}$. Next we equate $r_{B_2} =$ RHS of (35), by setting

$$\alpha_{B_2}^{(2)} = \frac{2^{r_{B_2}}\left(2^{r_{B_1}}+1\right)}{|h_{B_2R}|^2} \overset{(38),(41),|h_{B_2R}|^2\geq\frac{5}{2}}{\leq} 1, \tag{74}$$

verifying that this is a valid choice of $\alpha_{B_2}^{(2)}$. Then we equate $r_{A_2} - r_{B_2} =$ RHS of (33), by setting

$$\alpha_{A_2}^{(1)} = \frac{\left(2^{r_{A_2}-r_{B_2}}-1\right)\left(42^{r_{B_1}+r_{B_2}}+2\left(2^{r_{B_2}}+2^{r_{B_1}}\right)+1\right)}{|h_{A_2R}|^2}. \tag{75}$$

Using (23), $2^x + 2^y \leq 2^{x+y}$ with $x, y \geq 1$, and (74) and adding this to (75) we get

$$\begin{aligned}\alpha_{A_2}^{(1)} + \alpha_{A_2}^{(2)} =& \frac{\left(2^{r_{A_2}-r_{B_2}}-1\right)\left(42^{r_{B_1}+r_{B_2}}+2\left(2^{r_{B_2}}+2^{r_{B_1}}\right)+1\right)}{|h_{A_2R}|^2}+ \\ & \frac{\left(22^{r_{B_2}+r_{B_1}}+2^{r_{B_2}}\right)}{|h_{A_2R}|^2} \leq \frac{6\cdot 2^{r_{A_2}+r_{B_1}}+2^{r_{A_2}}-6}{|h_{A_2R}|^2} \overset{(38),(42)}{\leq} 1,\end{aligned} \tag{76}$$

verifying that this is a valid choice of $\alpha_{A_2}^{(1)}$, $\alpha_{A_2}^{(2)}$. Now we equate $r_{A_1} - r_{B_1} =$ RHS of (32), by setting

$$\alpha_{A_1}^{(1)} = \frac{\left(2^{r_{A_1}-r_{B_1}}-1\right)\left(42^{r_{B_2}+r_{B_1}}+2\left(2^{r_{B_1}}+2^{r_{B_2}}\right)+1\right)}{|h_{A_1R}|^2}. \tag{77}$$

Using (23), $2^x + 2^y \leq 2^{x+y}$ with $x, y \geq 1$, and (72) and adding this to (77) we get

$$\begin{aligned}\alpha_{A_1}^{(1)} + \alpha_{A_1}^{(2)} &\leq \frac{\left(2^{r_{A_1}-r_{B_1}}-1\right)\left(4\cdot 2^{r_{B_2}+r_{B_1}}+2\left(2^{r_{B_1}}+2^{r_{B_2}}\right)+1\right)}{|h_{A_1R}|^2} + \frac{2^{r_{B_1}}}{|h_{A_1R}|^2} \\ &\leq \frac{6\cdot 2^{r_{A_1}+r_{B_2}}+2^{r_{A_1}}-8}{|h_{A_1R}|^2} \overset{(37),(40)}{\leq} 1.\end{aligned} \tag{78}$$

which shows that this is a valid choice of $\alpha_{A_1}^{(1)}$, $\alpha_{A_1}^{(2)}$.

Finally we equate $r_{A_1} - r_{B_1} + r_{A_2} - r_{B_2} =$ RHS of (34), by setting

$$\alpha_{A_1}^{(1)} = \frac{\left(2^{r_{A_1}-r_{B_1}+r_{A_2}-r_{B_2}}-2^{r_{A_2}-r_{B_2}}\right)\left(4\cdot 2^{r_{B_2}+r_{B_1}}+2\left(2^{r_{B_1}}+2^{r_{B_2}}\right)+1\right)}{|h_{A_1R}|^2}. \tag{79}$$

Using (23), $2^x + 2^y \leq 2^{x+y}$ with $x, y \geq 1$, and (74) and adding this to (79) we get

$$\alpha_{A_1}^{(1)} + \alpha_{A_1}^{(2)} = \left(2^{r_{A_1} - r_{B_1} + r_{A_2} - r_{B_2}} - 2^{r_{A_2} - r_{B_2}}\right) \frac{\left(42^{r_{B_2} + r_{B_1}} + 2\left(2^{r_{B_1}} + 2^{r_{B_2}}\right) + 1\right)}{|h_{A_1R}|^2} + \frac{2^{r_{B_1}}}{|h_{A_1R}|^2} \quad (80)$$

$$\leq \frac{7 \cdot 2^{r_{A_1} + r_{A_2}} - 2}{|h_{A_1R}|^2} \overset{(39)}{\leq} 1.$$

which shows that this is a valid choice of $\alpha_{A_1}^{(1)}$, $\alpha_{A_1}^{(2)}$.

Appendix E
Decoding at the nodes

With $R_R^{(1)} = R_{A_1}^{(1)}$, $R_R^{(2)} = R_{A_1}^{(2)} = R_{B_1}$, $R_R^{(3)} = R_{A_2}^{(1)}$, $R_R^{(4)} = R_{A_2}^{(2)} = R_{B_2}$, we describe the decoding strategies at the nodes and the achievable rates for the case $|h_{RB_1}| \geq |h_{RA_1}| \geq |h_{RB_2}| \geq |h_{RA_2}|$.

A. *Decoding at node* B_1

The node B_1 first decodes $x_R^{(4)}$ (corresponds to f from the uplink) by treating $x_R^{(1)}$ to $x_R^{(3)}$ as noise. This can be done with low probability of error as long as

$$R_R^{(4)} \leq C\left(\frac{|h_{RB_1}|^2 \alpha_R^{(4)}}{1 + |h_{B_1R}|^2 \left(\sum_{j=1}^{3} \alpha_R^{(j)}\right)}\right) \quad (81)$$

Once decoded, the signal $x_R^{(4)}$ is canceled from the received signal and $x_R^{(3)}$ (corresponds to $x_{A_2}^{(1)}$ from the uplink) is decoded by treating $x_R^{(1)}$ and $x_R^{(2)}$ as noise. This can be done successfully with low probability of error as long as

$$R_R^{(3)} \leq C\left(\frac{|h_{RB_1}|^2 \alpha_R^{(3)}}{1 + |h_{B_1R}|^2 \left(\alpha_R^{(1)} + \alpha_R^{(2)}\right)}\right) \quad (82)$$

Once decoded, the signal $x_R^{(3)}$ is canceled from the received signal and $x_R^{(2)}$ (corresponds to t from the uplink) is decoded by treating $x_R^{(1)}$ as noise. This can be done successfully with low probability of error as long as

$$R_R^{(2)} \leq C\left(\frac{|h_{RB_1}|^2 \alpha_R^{(2)}}{1 + |h_{RB_1}|^2 \alpha_R^{(1)}}\right) \quad (83)$$

Once decoded, $x_R^{(2)}$ is canceled from the received signal. Finally, $x_R^{(1)}$ (corresponds to $x_{A_1}^{(1)}$ from the uplink) is decoded free of interference. This can be done with low probability of error as long as

$$R_R^{(1)} \leq C\left(|h_{RB_1}|^2 \alpha_R^{(1)}\right). \tag{84}$$

B. Decoding at node A_1

The node A_1 proceeds similarly with the exception that $x_R^{(1)}$ is known already and can be canceled from the received signal. After having decoded $x_R^{(3)}$ and $x_R^{(4)}$, $x_R^{(2)}$ is decoded free of interference. This can be done with low probability of error as long as

$$R_R^{(2)} \leq C\left(|h_{RA_1}|^2 \alpha_R^{(2)}\right). \tag{85}$$

C. Decoding at node B_2

The receivers of the second pair have the same order of detection. Thus, the node B_2 can decode $R_R^{(4)}$ with low probability of error as long as

$$R_R^{(4)} \leq C\left(\frac{|h_{RB_2}|^2 \alpha_R^{(4)}}{1 + |h_{RB_2}|^2 \left(\sum_{j=1}^{3} \alpha_R^{(j)}\right)}\right). \tag{86}$$

Once decoded, the signal $x_R^{(4)}$ is canceled from the received signal and $x_R^{(3)}$ is decoded by treating $x_R^{(1)}$ and $x_R^{(2)}$ as noise. This can be done successfully with low probability of error as long as

$$R_R^{(3)} \leq C\left(\frac{|h_{RB_2}|^2 \alpha_R^{(3)}}{1 + |h_{RB_2}|^2 \left(\alpha_R^{(1)} + \alpha_R^{(2)}\right)}\right) \tag{87}$$

D. Decoding at node A_2

Assuming that the node A_2 knows the strategy of the relay and the codebook it has used, it can reconstruct $x_R^{(3)}$ perfectly, since it contains only its own message. Thus, it cancels the effect of $x_R^{(3)}$ from the received signal. As a next and final step, it decodes $x_R^{(4)}$. This can be done with low probability of error as long as

$$R_R^{(4)} \leq C\left(\frac{|h_{RA_2}|^2 \alpha_R^{(4)}}{1 + |h_{RA_2}|^2 \left(\alpha_R^{(1)} + \alpha_R^{(2)}\right)}\right) \tag{88}$$

Thus, in summary we have

$$R_R^{(4)} \leq \min\left(\text{RHS of (86)}, \text{RHS of (88)}\right) \tag{89}$$

and

$$R_R^{(2)} \leq \min\left(\text{RHS of (85)}, \text{RHS of (83)}\right) \tag{90}$$

APPENDIX F
PROOF OF LEMMA 4

The three cases we have to consider are given in sections IV-E1-IV-E3. In the following we provide the proof for each case separately.

A. Case $|h_{RB_1}| \geq |h_{RA_1}| \geq |h_{RB_2}| \geq |h_{RA_2}|$

Consider a 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying (52)-(57). Starting with the first equation in (45), we equate

$$\log\left(1 + \alpha_R^{(1)}|h_{RB_1}|^2\right) = r_{A_1} - r_{B_1} \Rightarrow \alpha_R^{(1)} = \frac{2^{r_{A_1} - r_{B_1}} - 1}{|h_{RB_1}|^2}. \tag{91}$$

Now from (52) we know that

$$\alpha_R^{(1)} \leq \frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} \leq 1, \tag{92}$$

which shows that this is a valid choice of $\alpha_R^{(1)}$.

From (43), we have

$$R_{A_1}^{(2)}, R_{B_1} \leq C\left(\frac{|h_{RB_1}|^2\alpha_R^{(2)}}{1 + |h_{RB_1}|^2\alpha_R^{(1)}}\right). \tag{93}$$

Next we equate $r_{B_1} =$ RHS of (93), by setting

$$\alpha_R^{(2)} = \frac{\left(2^{r_{B_1}} - 1\right)\left(2^{r_{A_1} - r_{B_1}}\right)}{|h_{RB_1}|^2}. \tag{94}$$

Using (52) and (91) and adding this to (94) we get

$$\alpha_R^{(1)} + \alpha_R^{(2)} \leq 2\frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} \leq 1, \tag{95}$$

verifying that this is a valid choice of $\alpha_R^{(1)}$, $\alpha_R^{(2)}$. Then we equate $r_{A_2} - r_{B_2}$ = RHS of (45) (second equation), by setting

$$\alpha_R^{(3)} = \frac{(2^{r_{A_2}-r_{B_2}} - 1)\left(1 + \frac{|h_{RB_2}|^2}{|h_{RB_1}|^2}(2^{r_{B_2}} - 1)\right)}{|h_{RB_2}|^2}. \tag{96}$$

Using (53), (54) and (95) and adding this to (96) we get

$$\sum_{j=1}^{3} \alpha_R^{(j)} \leq \frac{\frac{1+|h_{RB_2}|^2}{4} - 1}{|h_{RB_2}|^2} + 3\frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} \leq 1, \tag{97}$$

verifying that this is a valid choice of $\alpha_R^{(j)}$, $j = 1 \ldots 3$. Finally from (44), we have

$$R_{A_2}^{(2)}, R_{B_2} \leq C\left(\frac{|h_{RA_2}|^2 \alpha_R^{(4)}}{1 + |h_{RA_2}|^2\left(\alpha_R^{(1)} + \alpha_R^{(2)}\right)}\right). \tag{98}$$

Thus, we equate r_{B_2} = RHS of (98), by setting

$$\alpha_R^{(4)} = \frac{(2^{r_{B_2}} - 1)}{|h_{RA_2}|^2}\left(1 + \frac{|h_{RA_2}|^2}{|h_{RB_1}|^2}(2^{r_{A_1}} - 1)\right). \tag{99}$$

Using (53), (55), (91), (94), (96) and adding this to (99) we get

$$\sum_{j=1}^{4} \alpha_R^{(j)} \leq \frac{\frac{1+|h_{RA_2}|^2}{4} - 1}{|h_{RA_2}|^2} + \frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} + \frac{\frac{1+|h_{RB_2}|^2}{4} - 1}{|h_{RB_2}|^2} + \frac{\frac{1+|h_{RB_2}|^2}{4} - 1}{|h_{RB_1}|^2} \leq 1 \tag{100}$$

which shows that this is a valid choice of $\alpha_R^{(j)}$, $j = 1 \ldots 4$.

B. *Case* $|h_{RB_1}| \geq |h_{RB_2}| \geq |h_{RA_1}| \geq |h_{RA_2}|$

Consider a 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying (52)-(57). Starting with the first equation in (48), we equate

$$\log\left(1 + \alpha_R^{(1)}|h_{RB_1}|^2\right) = r_{A_1} - r_{B_1} \Rightarrow \alpha_R^{(1)} = \frac{2^{r_{A_1}-r_{B_1}} - 1}{|h_{RB_1}|^2}. \tag{101}$$

Now from (52) we know that

$$\alpha_R^{(1)} \leq \frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} \leq 1, \tag{102}$$

which shows that this is a valid choice of $\alpha_R^{(1)}$.

Next we equate $r_{A_2} - r_{B_2}$ = RHS of (48) (second equation), by setting

$$\alpha_R^{(3)} = \frac{(2^{r_{A_2}-r_{B_2}} - 1)\left(1 + \frac{|h_{RB_2}|^2}{|h_{RB_1}|^2}(2^{r_{A_1}-r_{B_1}} - 1)\right)}{|h_{RB_2}|^2}. \tag{103}$$

Using (53), (54) and (95) and adding this to (103) we get

$$\alpha_R^{(1)} + \alpha_R^{(3)} \leq \frac{\frac{1+|h_{RB_2}|^2}{4} - 1}{|h_{RB_2}|^2} + \frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} \leq 1, \tag{104}$$

verifying that this is a valid choice of $\alpha_R^{(1)}$, $\alpha_R^{(3)}$.

From (46), we have

$$R_{A_1}^{(2)}, R_{B_1} \leq C\left(\frac{|h_{RA_1}|^2\alpha_R^{(2)}}{1+|h_{RA_1}|^2\alpha_R^{(3)}}\right). \tag{105}$$

Next we equate $r_{B_1} =$ RHS of (105), by setting

$$\alpha_R^{(2)} = \frac{(2^{r_{B_1}} - 1)\left(1 + |h_{RA_1}|^2\alpha_R^{(3)}\right)}{|h_{RA_1}|^2}. \tag{106}$$

Using (52) and (101) and adding this to (106) we get

$$\sum_{j=1}^{3}\alpha_R^{(j)} \leq \frac{2^{r_{B_1}} - 1}{|h_{RA_1}|^2} + \frac{2^{r_{B_1}+r_{A_2}} - 1}{|h_{RB_2}|^2} + \frac{2^{r_{A_1}+r_{A_2}} - 1}{|h_{RB_1}|^2} \leq 1, \tag{107}$$

verifying that this is a valid choice of $\alpha_R^{(j)}$, $j = 1\ldots 3$. Finally from (44), we have

$$R_{A_2}^{(2)}, R_{B_2} \leq C\left(\frac{|h_{RA_1}|^2\alpha_R^{(4)}}{1+|h_{RA_1}|^2\left(\alpha_R^{(3)}+\alpha_R^{(2)}\right)}\right). \tag{108}$$

Thus, we equate $r_{B_2} =$ RHS of (108), by setting

$$\alpha_R^{(4)} = \frac{(2^{r_{B_2}} - 1)}{|h_{RA_1}|^2}\left(2^{r_{A_1}} + (2^{r_{A_1}} - 1)\,|h_{RA_1}|^2\alpha_R^{(3)}\right). \tag{109}$$

Using (53), (55), (101), (106), (103) and adding this to (109) we get

$$\sum_{j=1}^{4}\alpha_R^{(j)} \leq \frac{\frac{1+|h_{RA_1}|^2}{4} - 1}{|h_{RA_1}|^2} + \frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} + \frac{\frac{1+|h_{RB_2}|^2}{4} - 1}{|h_{RB_2}|^2} + \frac{\frac{1+|h_{RB_2}|^2}{4} - 1}{|h_{RB_1}|^2} \leq 1 \tag{110}$$

which shows that this is a valid choice of $\alpha_R^{(j)}$, $j = 1\ldots 4$.

C. *Case* $|h_{RB_1}| \geq |h_{RB_2}| \geq |h_{RA_2}| \geq |h_{RA_1}|$

Consider a 4-tuple $(r_{A_1}, r_{B_1}, r_{A_2}, r_{B_2})$ satisfying (52)-(57). Starting with the first equation in (48), we equate

$$\log\left(1 + \alpha_R^{(1)}|h_{RB_1}|^2\right) = r_{A_1} - r_{B_1} \Rightarrow \alpha_R^{(1)} = \frac{2^{r_{A_1}-r_{B_1}} - 1}{|h_{RB_1}|^2}. \tag{111}$$

Now from (52) we know that

$$\alpha_R^{(1)} \le \frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} \le 1, \tag{112}$$

which shows that this is a valid choice of $\alpha_R^{(1)}$.

Next we equate $r_{A_2} - r_{B_2} =$ RHS of (51) (second equation), by setting

$$\alpha_R^{(3)} = \frac{\left(2^{r_{A_2}-r_{B_2}} - 1\right)\left(1 + \frac{|h_{RB_2}|^2}{|h_{RB_1}|^2}\left(2^{r_{A_1}-r_{B_1}} - 1\right)\right)}{|h_{RB_2}|^2}. \tag{113}$$

Using (53), (54) and (95) and adding this to (113) we get

$$\alpha_R^{(1)} + \alpha_R^{(3)} \le \frac{\frac{1+|h_{RB_2}|^2}{4} - 1}{|h_{RB_2}|^2} + \frac{\frac{1+|h_{RB_1}|^2}{4} - 1}{|h_{RB_1}|^2} \le 1, \tag{114}$$

verifying that this is a valid choice of $\alpha_R^{(1)}$, $\alpha_R^{(3)}$.

From (50), we have

$$R_{A_2}^{(2)}, R_{B_2} \le C\left(\frac{|h_{RA_2}|^2\alpha_R^{(4)}}{1 + |h_{RA_2}|^2\alpha_R^{(1)}}\right). \tag{115}$$

Next we equate $r_{B_2} =$ RHS of (115), by setting

$$\alpha_R^{(4)} = \frac{\left(2^{r_{B_2}} - 1\right)\left(1 + |h_{RA_2}|^2\alpha_R^{(1)}\right)}{|h_{RA_2}|^2}. \tag{116}$$

Using (52) and (111) and adding this to (116) we get

$$\alpha_R^{(1)} + \alpha_R^{(3)} + \alpha_R^{(4)} \le \frac{2^{r_{B_2}} - 1}{|h_{RA_2}|^2} + \frac{2^{r_{A_2}-r_{B_2}} - 1}{|h_{RB_2}|^2} + \frac{2^{r_{A_1}+r_{B_1}} - 1}{|h_{RB_1}|^2} + \frac{2^{r_{A_1}+r_{B_2}} - 1}{|h_{RB_1}|^2} \le 1, \tag{117}$$

verifying that this is a valid choice of $\alpha_R^{(1)}$, $\alpha_R^{(3)}$, and $\alpha_R^{(4)}$. Finally from (44), we have

$$R_{A_1}^{(2)}, R_{B_1} \le C\left(\frac{|h_{RA_1}|^2\alpha_R^{(2)}}{1 + |h_{RA_1}|^2\left(\alpha_R^{(3)} + \alpha_R^{(4)}\right)}\right). \tag{118}$$

Thus, we equate $r_{B_2} =$ RHS of (118), by setting

$$\alpha_R^{(2)} = \frac{\left(2^{r_{B_1}} - 1\right)}{|h_{RA_1}|^2}\left(1 + \left(\alpha_R^{(3)} + \alpha_R^{(4)}\right)|h_{RA_1}|^2\right). \tag{119}$$

Using (53), (55), (111), (116), (113) and adding this to (119) we get

$$\sum_{j=1}^{4}\alpha_R^{(j)} \le \frac{2^{r_{B_1}} - 1}{|h_{RA_1}|^2} + \frac{2^{r_{B_1}+r_{B_2}} - 1}{|h_{RA_2}|^2} + \frac{2^{r_{B_1}+r_{A_2}} - 1}{|h_{RB_2}|^2} + \frac{2^{r_{A_1}+r_{B_2}} - 1}{|h_{RB_1}|^2} + \frac{2^{r_{A_1}+r_{A_2}} - 1}{|h_{RB_1}|^2} \le 1, \tag{120}$$

which shows that this is a valid choice of $\alpha_R^{(j)}$, $j = 1 \ldots 4$.